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Density Operators for Fermions

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Abstract

The mathematical methods that have been used to analyze the statistical properties of boson fields, and in particular the coherence of photons in quantum optics, have their counterparts for Fermi fields. The coherent states, the displacement operators, the P-representation, and the other operator expansions all possess surprisingly close fermionic analogues. These methods for describing the statistical properties of fermions are based upon a practical calculus of anti-commuting variables. They are used to calculate correlation functions and counting distributions for general systems of fermions.

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1 Introduction

The Pauli exclusion principle plays an essential role in describing the behavior of the particles, both simple and complex, that we now call fermions. It is known to play a key role in determining the structure of the most fundamental elements of matter. These are systems like atoms, in which the phase-space density of fermions, electrons in this case, is quite high. But when fermionic atoms move freely in space or even when they are trapped electromagnetically, their phase-space density is usually so low that the effects of the exclusion principle remain completely hidden. A number of recent developments, however, point to the possibility of achieving much higher densities of fermionic atoms both in electromagnetic traps and in free space.

The various methods of optical cooling that have been developed for atomic beams work as well for fermions as they do for bosons and produce beams with temperatures of the order of 100 μ°K. Cooling fermions evaporatively to still lower temperatures poses a problem that requires a less direct solution. Evaporative cooling becomes inefficient for fermions since the exclusion principle tends to suppress collisions of identical atoms. It may be implemented nonetheless by sympathetic means [1], *e.g.,* by cooling bosonic atoms at the same time, so that energy exchange still takes place freely. It seems possible thus that the realization of degenerate Fermi gases may become an important byproduct of Bose-Einstein condensation.

The detection methods that will be used in measurements on beams of cold fermionic atoms will be essentially the same as those now used on bosonic atoms cooled by optical or evaporative means. The measurements on bosons can be most conveniently described, in fact, by mathematical methods that were introduced in the context of quantum optics [2, 3].

Much of the work in quantum optics, we may recall, is couched in the language of coherent states, which are eigenstates of the photon annihilation operators. They contain an intrinsically indefinite number of quanta but can nonetheless be used as a basis for describing all states of the electromagnetic field. While pure coherent states are not physically attainable in bosonic systems with fixed numbers of particles, it likewise remains useful to describe boson fields in terms of suitably weighted superpositions and mixtures of coherent states. The weight functions associated with these combinations

may be regarded as quasi-probability densities in the spaces of coherent-state amplitudes. The function P in the coherent-state representation of the density operator [2, 3] plays this role; other quasi-probability densities including the Wigner function [3, 4] and the Q function [3, 4] play similarly convenient roles in representing the density operator.

In the case of fermion fields, the vacuum state is the only physically realizable eigenstate of the annihilation operators. It is possible, however, to define such eigenstates in a formal way and to put them to many of the same analytical uses as are made of the bosonic coherent states. Since fermion field variables anti-commute, their eigenvalues must, as noted by Schwinger [6], be anti-commuting numbers. Such numbers are Grassmann variables. They can be handled by means of the simple rules of Grassmann algebra [7], which we include here so that the calculations may be self-contained.

Within this context we formulate ways of expressing and evaluating a broad range of the correlation functions that are measured in experiments involving the counting of fermions. Central to this task is the expression of the quantum-mechanical density operator in terms of Grassmann variables. We develop a number of ways of doing that in general terms and present a detailed discussion of the density operators for chaotically excited fields. Included among the latter is a particularly useful Gaussian representation of the grand-canonical density operator for fermion fields. Having evaluated the statistically averaged correlation functions, we apply them to fermion counting experiments and illustrate their use in determining the counting distributions.

We find throughout this work that notwithstanding great mathematical differences, many close parallels can be established between the expressions evaluated for fermion fields and the more familiar ones for boson fields. In particular, for example, we can construct a family of quasi-probability densities, as functions of the Grassmann variables, with properties parallel to those of the entire family of quasi-probability densities for bosons, including the P, Q, and Wigner functions. We can then evaluate the mean values of ordered products of fermion creation and annihilation operators by performing integrations over the Grassmann variables while using the appropriate quasi-probability density as a weight function. In both cases, we trade an inhomogeneous commutation relation and an ordering rule for a homogeneous

commutation relation and a quasi-probability density. For boson fields the integrations are taken over commuting variables, which may be treated as if they were classical variables. For fermions, on the other hand, the integrations are over anti-commuting variables that have no classical analogues. The weight functions for these integrations are nevertheless in one-to-one correspondence with the quasi-probability densities for bosons, so it seems appropriate to give them similar names. We have followed that convention for several other parallels as well.

2 Notation

Let us consider a system of fermions which may be described by the creation $a_n^\dagger$ and annihilation a_m operators which satisfy the familiar but ever mysterious relations

$$\{a_n, a_m^\dagger\} = \delta_{nm} \tag{1}$$
$$\{a_n, a_m\} = 0 \tag{2}$$
$$\{a_n^\dagger, a_m^\dagger\} = 0 \tag{3}$$
$$a_n|0\rangle = 0. \tag{4}$$

in which $|0\rangle$ is the vacuum state.

We shall use lower-case Greek letters to denote Grassmann variables. These anti-commuting numbers γ_n and their complex conjugates γ_n^* satisfy the convenient relations

$$\{\gamma_n, \gamma_m\} = 0 \tag{5}$$
$$\{\gamma_n^*, \gamma_m\} = 0 \tag{6}$$
$$\{\gamma_n^*, \gamma_m^*\} = 0. \tag{7}$$

We shall also assume that Grassmann variables anti-commute with fermionic operators

$$\{\gamma_n, a_m\} = 0 \tag{8}$$

and commute with bosonic operators. And we make the arbitrary choice that hermitian conjugation reverses the order of all fermionic quantities, both the

operators and the Grassmann numbers. Thus for instance we have

$$\left(a_1\beta_2 a_3^\dagger \gamma_4^*\right)^\dagger = \gamma_4 a_3 \beta_2^* a_1^\dagger. \tag{9}$$

3 Coherent States for Fermions

3.1 Displacement Operators

For any set $\boldsymbol{\gamma} = \{\gamma_i\}$ of Grassmann variables, let us define the unitary displacement operator $D(\boldsymbol{\gamma})$ as the exponential

$$D(\boldsymbol{\gamma}) = \exp\left(\sum_i \left(a_i^\dagger \gamma_i - \gamma_i^* a_i\right)\right). \tag{10}$$

One of the useful properties of Grassmann numbers is that when, as in the preceding definition, they multiply fermionic annihilation or creation operators, their anti-commutativity cancels that of the operators. Thus the operators $a_i^\dagger \gamma_i$ and $\gamma_j^* a_j$ simply commute for $i \neq j$. So we may rewrite the displacement operator as the product

$$\begin{aligned} D(\boldsymbol{\gamma}) &= \prod_i \exp\left(a_i^\dagger \gamma_i - \gamma_i^* a_i\right) &(11)\\ &= \prod_i \left[1 + a_i^\dagger \gamma_i - \gamma_i^* a_i + \left(a_i^\dagger a_i - \tfrac{1}{2}\right)\gamma_i^*\gamma_i\right]. &(12) \end{aligned}$$

By the same token, the annihilation operator a_n commutes with all the operators $a_i^\dagger \gamma_i$ and $\gamma_j^* a_j$ when $n \neq i$, and so we may compute the displaced annihilation operator by ignoring all modes but the nth:

$$\begin{aligned} D^\dagger(\boldsymbol{\gamma}) a_n D(\boldsymbol{\gamma}) &= \prod_i \exp\left(\gamma_i^* a_i - a_i^\dagger \gamma_i\right) a_n \prod_j \exp\left(a_j^\dagger \gamma_j - \gamma_j^* a_j\right) \\ &= \exp\left(\gamma_n^* a_n - a_n^\dagger \gamma_n\right) a_n \exp\left(a_n^\dagger \gamma_n - \gamma_n^* a_n\right) \\ &= \left(1 - a_n^\dagger \gamma_n - \tfrac{1}{2}\gamma_n^* a_n a_n^\dagger \gamma_n\right) a_n \left(1 + a_n^\dagger \gamma_n - \tfrac{1}{2} a_n^\dagger \gamma_n \gamma_n^* a_n\right) \\ &= \left(1 - a_n^\dagger \gamma_n - \tfrac{1}{2}\gamma_n^* \gamma_n\right) a_n \left(1 + a_n^\dagger \gamma_n + \tfrac{1}{2}\gamma_n^* \gamma_n\right) \\ &= a_n - a_n^\dagger \gamma_n a_n + a_n a_n^\dagger \gamma_n = a_n + \gamma_n. &(13) \end{aligned}$$

Similarly

$$D^\dagger(\boldsymbol{\gamma}) a_n^\dagger D(\boldsymbol{\gamma}) = a_n^\dagger + \gamma_n^*. \tag{14}$$

We may use the Baker-Hausdorff identity

$$e^{A+B} = e^A \, e^B \, e^{-\frac{1}{2}[A,B]}, \tag{15}$$

which holds whenever the commutator $[A, B]$ commutes with both A and B, to write the displacement operator $D(\boldsymbol{\alpha})$ in forms that are normally ordered

$$\begin{aligned} \exp\left(\sum_i a_i^\dagger \gamma_i\right) \exp\left(-\sum_i \gamma_i^* a_i\right) &= \exp\left(\sum_i \left(a_i^\dagger \gamma_i - \gamma_i^* a_i\right)\right) e^{\frac{1}{2}\boldsymbol{\gamma}^* \cdot \boldsymbol{\gamma}} \\ D_N(\boldsymbol{\gamma}) &= D(\boldsymbol{\gamma}) \exp\left(\tfrac{1}{2} \textstyle\sum_i \gamma_i^* \gamma_i\right) \end{aligned} \tag{16}$$

and anti-normally ordered

$$\begin{aligned} \exp\left(-\sum_i \gamma_i^* a_i\right) \exp\left(\sum_i a_i^\dagger \gamma_i\right) &= \exp\left(\sum_i \left(a_i^\dagger \gamma_i - \gamma_i^* a_i\right)\right) e^{-\frac{1}{2}\boldsymbol{\gamma}^* \cdot \boldsymbol{\gamma}} \\ D_A(\boldsymbol{\gamma}) &= D(\boldsymbol{\gamma}) \exp\left(-\tfrac{1}{2} \textstyle\sum_i \gamma_i^* \gamma_i\right), \end{aligned} \tag{17}$$

in which we have employed the concise notation

$$\boldsymbol{\gamma}^* \cdot \boldsymbol{\gamma} \equiv \sum_i \gamma_i^* \gamma_i, \tag{18}$$

an abbreviation which we shall use occasionally but not exclusively. The identity (15) also allows one to show that the displacement operators form a ray representation of the additive group of Grassmann numbers,

$$D(\boldsymbol{\alpha}) \, D(\boldsymbol{\beta}) = D(\boldsymbol{\alpha} + \boldsymbol{\beta}) \exp\left[\tfrac{1}{2} \textstyle\sum_i \left(\beta_i^* \alpha_i - \alpha_i^* \beta_i\right)\right]. \tag{19}$$

3.2 Coherent States

For any set $\boldsymbol{\gamma} = \{\gamma_i\}$ of Grassmann numbers, we define the *normalized* coherent state $|\boldsymbol{\gamma}\rangle$ as the displaced vacuum state

$$|\boldsymbol{\gamma}\rangle = D(\boldsymbol{\gamma})|0\rangle. \tag{20}$$

By using the displacement relation (13), we may show that the coherent state is an eigenstate of every annihilation operator a_n:

$$\begin{aligned} a_n|\boldsymbol{\gamma}\rangle &= a_n D(\boldsymbol{\gamma})|0\rangle = D(\boldsymbol{\gamma})\, D^\dagger(\boldsymbol{\gamma}) a_n D(\boldsymbol{\gamma})|0\rangle \\ &= D(\boldsymbol{\gamma})\ (a_n + \gamma_n)\,|0\rangle = D(\boldsymbol{\gamma})\,\gamma_n|0\rangle = \gamma_n\, D(\boldsymbol{\gamma})|0\rangle \\ &= \gamma_n|\boldsymbol{\gamma}\rangle. \end{aligned} \tag{21}$$

By using the product formula (12) for the displacement operator, we may write the coherent state in the form

$$\begin{aligned} |\boldsymbol{\gamma}\rangle &= D(\boldsymbol{\gamma})|0\rangle = \prod_i \left[1 + a_i^\dagger\gamma_i - \gamma_i^* a_i + \left(a_i^\dagger a_i - \tfrac{1}{2}\right)\gamma_i^*\gamma_i\right]|0\rangle \\ &= \prod_i \left(1 + a_i^\dagger\gamma_i - \tfrac{1}{2}\gamma_i^*\gamma_i\right)|0\rangle \\ &= \exp\left(\sum_i \left(a_i^\dagger\gamma_i - \tfrac{1}{2}\gamma_i^*\gamma_i\right)\right)|0\rangle. \end{aligned} \tag{22}$$

It may be worth emphasizing that in this formula the creation operator $a_i^\dagger$ stands to the *left* of the Grassmann number γ_i. Apart from these ordering considerations, this formula takes a form closely analogous to the one that defines bosonic coherent states.

The adjoint of the coherent state $|\boldsymbol{\gamma}\rangle$ is

$$\langle\boldsymbol{\gamma}| = \langle 0|D^\dagger(\boldsymbol{\gamma}) = \langle 0|\exp\left(\sum_i \left(\gamma_i^* a_i - \tfrac{1}{2}\gamma_i^*\gamma_i\right)\right), \tag{23}$$

and it obeys the relation

$$\langle\boldsymbol{\gamma}|a_n^\dagger = \langle\boldsymbol{\gamma}|\gamma_n^*. \tag{24}$$

The inner product of two coherent states is

$$\langle\boldsymbol{\gamma}|\boldsymbol{\beta}\rangle = \exp\left(\sum_i \left(\gamma_i^*\beta_i - \tfrac{1}{2}(\gamma_i^*\gamma_i + \beta_i^*\beta_i)\right)\right), \tag{25}$$

so that

$$\begin{aligned} \langle\boldsymbol{\beta}|\boldsymbol{\gamma}\rangle\,\langle\boldsymbol{\gamma}|\boldsymbol{\beta}\rangle &= \exp\left[-\sum_i (\beta_i^* - \gamma_i^*)(\beta_i - \gamma_i)\right] \\ &= \prod_i \left[1 - (\beta_i^* - \gamma_i^*)(\beta_i - \gamma_i)\right]. \end{aligned} \tag{26}$$

In contrast to the case of bosons, we may for fermions define for any set $\boldsymbol{\alpha} = \{\alpha_i\}$ of Grassmann numbers the normalized eigenstate $|\boldsymbol{\alpha}\rangle'$ of the fermion creation operators $a_i^\dagger$ as the displaced state

$$|\boldsymbol{\alpha}\rangle' = D(\boldsymbol{\alpha})|\mathbf{1}\rangle \tag{27}$$

where $|\mathbf{1}\rangle$ is the state in which every mode is filled:

$$|\mathbf{1}\rangle = \prod_n a_n^\dagger |0\rangle. \tag{28}$$

By using the displacement relation (14), we may show that the state $|\boldsymbol{\alpha}\rangle'$ is an eigenstate of every creation operator $a_n^\dagger$:

$$\begin{aligned} a_n^\dagger |\boldsymbol{\alpha}\rangle' &= a_n^\dagger D(\boldsymbol{\alpha})|\mathbf{1}\rangle = D(\boldsymbol{\alpha})\, D^\dagger(\boldsymbol{\alpha}) a_n^\dagger D(\boldsymbol{\alpha})|\mathbf{1}\rangle \\ &= D(\boldsymbol{\alpha}) \left(a_n^\dagger + \alpha_n^*\right)|\mathbf{1}\rangle = D(\boldsymbol{\alpha})\, \alpha_n^* |\mathbf{1}\rangle = \alpha_n^*\, D(\boldsymbol{\alpha})|\mathbf{1}\rangle \\ &= \alpha_n^* |\boldsymbol{\alpha}\rangle'. \end{aligned} \tag{29}$$

The adjoint relation is

$${}'\langle\boldsymbol{\alpha}| a_n = {}'\langle\boldsymbol{\alpha}|\alpha_n. \tag{30}$$

An explicit formula for the eigenstate $|\boldsymbol{\alpha}\rangle'$ follows from its definition (27):

$$|\boldsymbol{\alpha}\rangle' = \prod_i \left(1 - \alpha_i^* a_i + \tfrac{1}{2}\alpha_i^* \alpha_i\right)|\mathbf{1}\rangle. \tag{31}$$

3.3 Intrinsic Descriptions of Fermionic States

The occupation-number description of states of fermions has well-known ambiguities. For $n \neq m$, for example, the state $|1_n 1_m\rangle$ may be interpreted as $a_n^\dagger a_m^\dagger |0\rangle$ or as $a_m^\dagger a_n^\dagger |0\rangle = -a_n^\dagger a_m^\dagger |0\rangle$.

The creation operators themselves provide an unambiguous description of fermionic states,

$$|\psi\rangle = \sum_{\{n\}} c(n_1, n_2, \ldots)\, a_{n_1}^\dagger a_{n_2}^\dagger \ldots a_{n_m}^\dagger\, |0\rangle, \tag{32}$$

which transfers to the coherent-state representation

$$\langle\boldsymbol{\alpha}|\psi\rangle = \exp\left(-\tfrac{1}{2}\textstyle\sum_n \alpha_n^* \alpha_n\right) \displaystyle\sum_{\{n\}} c(n_1, n_2, \ldots)\, \alpha_{n_1}^* \alpha_{n_2}^* \ldots \alpha_{n_m}^* \tag{33}$$

without any ambiguity or extra minus signs. Because coherent states are defined in terms of bilinear forms in anti-commuting variables, there is no need to adopt a standard ordering of the modes.

4 Grassmann Calculus

4.1 Differentiation

Since the square of any Grassmann variable vanishes, the most general function $f(\xi)$ of a single anti-commuting variable ξ is linear in ξ

$$f(\xi) = u + \xi t. \tag{34}$$

We define the left derivative of the function $f(\xi)$ with respect to the Grassmann variable ξ as

$$\frac{df(\xi)}{d\xi} = t. \tag{35}$$

Note that if the variable t is anti-commuting, then we may also write the function $f(\xi)$ in the form

$$f(\xi) = u - t\xi. \tag{36}$$

Now to form the left derivative, we first move ξ past t, picking up a minus sign and obtaining the form (34) and the result (35). In this case, the right derivative is $-t$. In the present work, we shall use left derivatives exclusively and shall refer to them simply as derivatives.

4.2 Even and Odd Functions

It is useful to distinguish between functions that commute with Grassmann variables and ones that do not. We shall say that a function $f(\boldsymbol{\alpha})$ that commutes with Grassmann variables is *even* and that a function $f(\boldsymbol{\alpha})$ that anti-commutes with Grassmann variables is *odd*. We shall often note the evenness or oddness of the functions we introduce.

4.3 Product Rule

To compute the derivative of the product of two functions $f(\boldsymbol{\alpha})$ and $g(\boldsymbol{\alpha})$ with respect to a particular variable α_i, one may explicitly move the α_i in $g(\boldsymbol{\alpha})$ through $f(\boldsymbol{\alpha})$ or one may move the operator representing differentiation through the function $f(\boldsymbol{\alpha})$. In either case if the function $f(\boldsymbol{\alpha})$ is odd, then one picks up a minus sign. The product rule is thus

$$\frac{\partial}{\partial \alpha_i} \left(f(\boldsymbol{\alpha})\, g(\boldsymbol{\alpha}) \right) = \frac{\partial f(\boldsymbol{\alpha})}{\partial \alpha_i}\, g(\boldsymbol{\alpha}) + \sigma(f)\, f(\boldsymbol{\alpha})\, \frac{\partial g(\boldsymbol{\alpha})}{\partial \alpha_i}, \tag{37}$$

where the sign $\sigma(f)$ of $f(\boldsymbol{\alpha})$ is -1 if $f(\boldsymbol{\alpha})$ is an odd function and $+1$ if $f(\boldsymbol{\alpha})$ is even.

4.4 Integration

We define a sort of integration over the complex Grassmann variables by the following rules

$$\int d\alpha_n = \int d\alpha_n^* = 0 \tag{38}$$

$$\int d\alpha_n \, \alpha_m = \delta_{nm} \tag{39}$$

$$\int d\alpha_n^* \, \alpha_m^* = \delta_{nm}. \tag{40}$$

This integration due to Berezin [7] is exactly equivalent to left differentiation.

We shall typically be concerned with pairs of anti-commuting variables α_i and α_i^*, and for such pairs we shall adhere to the notation

$$\int d^2\alpha_n = \int d\alpha_n^* \, d\alpha_n \tag{41}$$

in which the differential of the conjugated variable comes first. Note that

$$d\alpha_n \, d\alpha_n^* = -d\alpha_n^* \, d\alpha_n. \tag{42}$$

We have been using boldface type to denote sets of Grassmann variables; we shall extend that use to write multiple integrals over such sets in the succinct form

$$\int d^2\boldsymbol{\alpha} \equiv \int \prod_i d^2\alpha_i. \tag{43}$$

We shall also occasionally employ the concise notation

$$\boldsymbol{\alpha^*} \cdot \boldsymbol{\beta} \equiv \sum_n \alpha_n^* \beta_n \tag{44}$$

for sums of simple products over all the modes of the system.

The simple integral formula

$$\begin{aligned} \int d^2\alpha_n \, e^{-\alpha_p^*\alpha_q} &= \int d\alpha_n^* \, d\alpha_n \left(1 - \alpha_p^*\alpha_q\right) = -\int d\alpha_n^* \, d\alpha_n \, \alpha_p^*\alpha_q \\ &= \int d\alpha_n^* \, \alpha_p^* \, d\alpha_n \, \alpha_q = \int d\alpha_n^* \, \alpha_p^* \, \delta_{nq} = \delta_{np} \, \delta_{nq} \end{aligned} \tag{45}$$

provides a useful example of Grassmann integration. We also note the general rule

$$\int d^2\alpha \, f(\lambda\alpha) = |\lambda|^2 \int d^2\beta \, f(\beta), \tag{46}$$

in which λ is an arbitrary complex number and in which $f(\lambda\alpha)$ is an abbreviation for a function which necessarily depends on both $\lambda\alpha$ and $\lambda^*\alpha^*$. This rule owes its strange appearance to the definition of integration as differentiation.

Some further examples are the integral of the exponential function

$$\int d^2\alpha \, \exp\left(\beta^*\alpha + \alpha^*\gamma + \alpha\alpha^*\right) = \exp\left(\beta^*\gamma\right) \tag{47}$$

and the Fourier transform of a gaussian

$$\int d^2\xi \, \exp\left(\alpha\xi^* - \xi\alpha^* + \lambda\xi\xi^*\right) = \lambda \, \exp\left(\frac{\alpha\alpha^*}{\lambda}\right) \tag{48}$$

where λ is an arbitrary complex number. The latter integral can be written in a somewhat more-general form which is no longer a Fourier transform:

$$\int d^2\xi \, \exp\left(\alpha\xi^* - \xi\beta^* + \lambda\xi\xi^*\right) = \lambda \, \exp\left(\frac{\alpha\beta^*}{\lambda}\right). \tag{49}$$

4.5 Integration by Parts

Let us first observe that the integral of a derivative vanishes

$$\int d^2\boldsymbol{\alpha}\, \frac{\partial f(\boldsymbol{\alpha})}{\partial \alpha_i} = 0 \tag{50}$$

because the derivative with respect to the variable α_i lacks the variable α_i. In particular the integral of the derivative of the product of two functions also vanishes, and so by using the product rule (37), we have

$$\int d^2\boldsymbol{\alpha}\, \frac{\partial}{\partial \alpha_i}\left(f(\boldsymbol{\alpha})\, g(\boldsymbol{\alpha})\right) = \int d^2\boldsymbol{\alpha}\, \left[\left(\frac{\partial f(\boldsymbol{\alpha})}{\partial \alpha_i}\right) g(\boldsymbol{\alpha}) + \sigma(f)\, f(\boldsymbol{\alpha}) \frac{\partial g(\boldsymbol{\alpha})}{\partial \alpha_i}\right] = 0, \tag{51}$$

which is the formula for integration by parts,

$$\int d^2\boldsymbol{\alpha}\, \left(\frac{\partial f(\boldsymbol{\alpha})}{\partial \alpha_i}\right) g(\boldsymbol{\alpha}) = -\sigma(f) \int d^2\boldsymbol{\alpha}\, f(\boldsymbol{\alpha})\, \frac{\partial g(\boldsymbol{\alpha})}{\partial \alpha_i}, \tag{52}$$

where the sign $\sigma(f)$ is $+1$ if the function $f(\boldsymbol{\alpha})$ is even and -1 if it is odd.

4.6 Completeness of the Coherent States

We may use our Grassmann calculus to show that the coherent states are complete. Let us consider the state

$$|f\rangle = \left(c + d a^\dagger\right)|0\rangle, \tag{53}$$

which for arbitrary complex numbers c and d is an arbitrary single-mode state. Then its inner product $\langle\gamma|f\rangle$ with the coherent state $|\gamma\rangle$ is the correct weight function for the coherent-state expansion since

$$\int d^2\gamma \langle\gamma|f\rangle|\gamma\rangle = \int d^2\gamma (c + d\gamma^*)\,\left(1 + \gamma\gamma^* - \gamma a^\dagger\right)|0\rangle = \left(c + d a^\dagger\right)|0\rangle = |f\rangle. \tag{54}$$

The reader may generalize this example to the multi-mode case. The coherent states in fact are over-complete.

4.7 Completeness of the Displacement Operators

For a single mode, the identity operator I and the traceless operators $a, a^\dagger$, and $\frac{1}{2} - a^\dagger a$ form a complete set of operators. Since by using the expression (12) and our Grassmann calculus, we may write each of these operators as an integral over the displacement operators

$$\begin{aligned} I &= \int d^2\gamma\, \gamma\gamma^*\, D(\gamma) && (55)\\ a &= \int d^2\gamma\, (-\gamma)\, D(\gamma) && (56)\\ a^\dagger &= \int d^2\gamma\, \gamma^*\, D(\gamma) && (57)\\ \tfrac{1}{2} - a^\dagger a &= \int d^2\gamma\, D(\gamma), && (58) \end{aligned}$$

it follows that the displacement operators form a complete set of operators for that mode. It is easy to generalize this proof to the multi-mode case. The displacement operators are over-complete.

5 Operators

Some operators can be written as sums of products of even numbers of creation and annihilation operators; we shall call such operators *even.* Operators that can be written as sums of products of odd numbers of creation and annihilation operatorswe shall shall speak of as *odd.* Although most operators are neither even nor odd, the operators of physical interest are either even or odd. The number operator $a^\dagger a$, for example, is even, while the creation and annihilation operators, $a^\dagger$ and a, are odd.

The operators of quantum mechanics and of quantum field theory do not themselves involve Grassmann variables. Thus even operators commute with Grassmann variables, while odd ones anti-commute.

5.1 The Identity Operator

If we compare the integral

$$\int d^2\boldsymbol{\alpha}\,|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle = \int d^2\boldsymbol{\alpha}\,\exp\left(\sum_i\left(a_i^\dagger\alpha_i + \alpha_i^*\beta_i + \alpha_i\alpha_i^* + \tfrac{1}{2}\beta_i\beta_i^*\right)\right)|0\rangle \tag{59}$$

with the integral-formula (47) and identify β^* and γ in that formula with $a^\dagger$ and β in this integral, then we have

$$\int d^2\boldsymbol{\alpha}\,|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle = \exp\left(\sum_i\left(a_i^\dagger\beta_i + \tfrac{1}{2}\beta_i\beta_i^*\right)\right)|0\rangle = |\boldsymbol{\beta}\rangle. \tag{60}$$

Since the coherent states form a complete set of states, as shown by the expansion (54), it follows that the identity operator is given by the integral

$$I = \int d^2\boldsymbol{\alpha}\,|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|. \tag{61}$$

The corresponding expression for the identity operator in terms of the eigenstates $|\boldsymbol{\alpha}\rangle'$ of the creation operators is

$$I = \int\prod_i\left(-d^2\alpha_i\right)\,|\boldsymbol{\alpha}\rangle'\,'\langle\boldsymbol{\alpha}|. \tag{62}$$

5.2 The Trace

The trace of an arbitrary operator B is the sum of the diagonal matrix elements of B in the n-quantum states,

$$\mathrm{Tr}B = \sum_n\langle n|B|n\rangle, \tag{63}$$

which shows that the trace of an operator that is odd vanishes. By inserting the preceding formula (61) for the identity operator, we have

$$\mathrm{Tr}B = \sum_n\int d^2\alpha\,\langle n|\alpha\rangle\langle\alpha|B|n\rangle. \tag{64}$$

If we move the coherent-state matrix element $\langle n|\alpha\rangle$ to the right of the matrix element of the even operator B, then we see from the formula (22) that minus signs arise that can be absorbed into the argument of either of the two coherent states,

$$\mathrm{Tr}B = \sum_n \int d^2\alpha \, \langle\alpha|B|n\rangle\langle n| - \alpha\rangle = \sum_n \int d^2\alpha \, \langle -\alpha|B|n\rangle\langle n|\alpha\rangle, \tag{65}$$

in which the sum $\sum_n |n\rangle\langle n| = I$ is the identity operator. The resulting multi-mode trace formula is

$$\mathrm{Tr}B = \int d^2\boldsymbol{\alpha} \, \langle\boldsymbol{\alpha}|B| - \boldsymbol{\alpha}\rangle = \int d^2\boldsymbol{\alpha} \, \langle -\boldsymbol{\alpha}|B|\boldsymbol{\alpha}\rangle, \tag{66}$$

which holds also for odd operators, both sides vanishing. An important example is the trace of the dyadic operator $|\boldsymbol{\beta}\rangle\langle\boldsymbol{\gamma}|$,

$$\begin{aligned} \mathrm{Tr}|\boldsymbol{\beta}\rangle\langle\boldsymbol{\gamma}| &= \int d^2\boldsymbol{\alpha} \, \langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle\langle\boldsymbol{\gamma}| - \boldsymbol{\alpha}\rangle = \int d^2\boldsymbol{\alpha} \, \langle\boldsymbol{\gamma}| - \boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle \\ &= \int d^2\boldsymbol{\alpha} \, \langle -\boldsymbol{\gamma}|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle = \langle -\boldsymbol{\gamma}|\boldsymbol{\beta}\rangle = \langle\boldsymbol{\gamma}| - \boldsymbol{\beta}\rangle, \end{aligned} \tag{67}$$

in which we have used the completeness relation (61). Since the coherent states are complete, we may replace in this formula either the ket $|\boldsymbol{\beta}\rangle$ or the bra $\langle\boldsymbol{\gamma}|$ with its image $F|\boldsymbol{\beta}\rangle$ or $\langle\boldsymbol{\gamma}|F$ under the action of the arbitrary operator F and obtain the trace formula

$$\mathrm{Tr}\left(F|\boldsymbol{\beta}\rangle\langle\boldsymbol{\gamma}|\right) = \mathrm{Tr}\left(|\boldsymbol{\beta}\rangle\langle\boldsymbol{\gamma}|F\right) = \langle -\boldsymbol{\gamma}|F|\boldsymbol{\beta}\rangle = \langle\boldsymbol{\gamma}|F| - \boldsymbol{\beta}\rangle. \tag{68}$$

5.3 Physical States and Operators

A state $|\psi\rangle$ is *physical* if it changes at most by a phase when subjected to a rotation of angle 2π about any axis,

$$U(\hat{n}, 2\pi)|\psi\rangle = e^{i\theta}\,|\psi\rangle. \tag{69}$$

Since fermions carry half-odd-integer spin, a state of one fermion or of any odd number of fermions changes by the phase factor -1. States that contain no fermions or only even numbers of fermions are invariant under such 2π rotations.

Thus physical states are linear combinations of states with odd numbers of fermions or linear combinations of states with even numbers of fermions. But a state that is a linear combination of a state that contains an odd number of fermions and another that contains an even number of fermions is excluded. For instance, the state

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \tag{70}$$

is unphysical because under a 2π rotation it changes into a different state:

$$U(\hat{n}, 2\pi)\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \neq e^{i\theta}\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right). \tag{71}$$

We define an operator as *physical* if it maps physical states onto physical states. Physical operators are either even or odd.

In all physical contexts that have been explored experimentally, the number of fermions (or more generally the number of fermions minus the number of anti-fermions) is strictly conserved. That conservation law leads to certain further restrictions on the permissible states of the field. If we let $N = \sum_k a_k^\dagger a_k$ be the fermion number, the law requires that any state arising from an eigenstate of N must remain an eigenstate of N. This law can be derived from an assumed $U(1)$ invariance of all the interactions under the transformation $U(\theta) = \exp(i\theta N)$, which changes a and $a^\dagger$ to

$$e^{-i\theta N} a e^{i\theta N} = e^{i\theta} a \tag{72}$$

and

$$e^{-i\theta N} a^\dagger e^{i\theta N} = e^{-i\theta} a^\dagger. \tag{73}$$

Fermion conserving interactions involving the a_k and $a_k^\dagger$ are ones in which the phase factors $e^{\pm i\theta}$ all cancel. If a system begins in a state with a fixed number of fermions, the conservation law restricts the set of accessible states considerably more than the 2π super-selection rule mentioned earlier. Transitions can not be made, for example, between states with different even fermion numbers or between states with different odd fermion numbers.

5.4 Physical Density Operators

A physical density operator can be written as a sum of dyadics of physical states with positive coefficients that add up to unity. It follows that a physical density operator ρ is a positive hermitian operator of unit trace: for any state $|\psi\rangle$

$$\begin{aligned} \langle\psi|\rho|\psi\rangle &\geq 0 && (74) \\ \rho^\dagger &= \rho && (75) \\ \mathrm{Tr}\rho &= 1. && (76) \end{aligned}$$

Physical density operators are invariant under a 2π rotation. Thus the one-mode operator

$$\rho = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right), \tag{77}$$

for example, is a physical density operator, but the dyadic

$$\frac{1}{2}\left(|0\rangle + |1\rangle\right)\left(\langle 0| + \langle 1|\right) \tag{78}$$

is not. In this work we shall consider only density operators that are physical in this sense.

The dynamical problems we solve do not always begin with a fixed number of fermions. More generally they begin with a mixture of states with different fermion numbers, that is with density operators of the form

$$\rho = \sum_{N'} p_{N'}\, |N'\rangle\langle N'| \tag{79}$$

where the $p_{N'}$ are real and non-negative. Such density operators are invariant under the transformation $U(\theta) = \exp(i\theta N)$, and the fermion conservation law assures us that they will always remain so,

$$e^{-i\theta N}\rho e^{i\theta N} = \rho. \tag{80}$$

The coherent states do undergo a simple change under this transformation,

$$U(\theta)|\alpha\rangle = e^{i\theta N}|\alpha\rangle = |e^{i\theta}\alpha\rangle \tag{81}$$

$$\langle\alpha|U^\dagger(\theta) = \langle\alpha|e^{-i\theta N} = \langle e^{i\theta}\alpha|, \tag{82}$$

which leaves their scalar product invariant,

$$\langle e^{i\theta}\alpha|e^{i\theta}\alpha\rangle = \langle\alpha|\alpha\rangle. \tag{83}$$

6 Delta Functions and Fourier Transforms

We can define a function

$$\delta(\boldsymbol{\xi} - \boldsymbol{\zeta}) \equiv \int d^2\boldsymbol{\alpha} \exp\left(\sum_n \left(\alpha_n\left(\xi_n^* - \zeta_n^*\right) - \left(\xi_n - \zeta_n\right)\alpha_n^*\right)\right) \tag{84}$$

$$= \prod_n \left(\xi_n - \zeta_n\right)\left(\xi_n^* - \zeta_n^*\right) \tag{85}$$

which plays the role of a Dirac delta function in that if $f(\boldsymbol{\xi})$ is any function of the set $\boldsymbol{\xi}$ of Grassmann variables $\{\xi_1, \xi_2, \ldots\}$, then

$$\int d^2\boldsymbol{\xi}\, \delta(\boldsymbol{\xi} - \boldsymbol{\zeta})\, f(\boldsymbol{\xi}) = f(\boldsymbol{\zeta}). \tag{86}$$

The delta function is doubly even: it commutes with Grassmann numbers and $\delta(\xi - \zeta) = \delta(\zeta - \xi)$.

We have been using the term Fourier transform to denote an integral of the form

$$\tilde{f}(\alpha) = \int d^2\xi\, e^{\alpha\xi^* - \xi\alpha^*} f(\xi). \tag{87}$$

The delta-function identity (84) implies that the inverse Fourier transform is given by the similar formula

$$f(\xi) = \int d^2\alpha\, e^{\xi\alpha^* - \alpha\xi^*} \tilde{f}(\alpha). \tag{88}$$

The identity (84) also leads to two forms of Parseval's relation:

$$\int d^2\alpha\, \tilde{f}(\alpha)\, \left[\tilde{g}(\alpha)\right]^* = \int d^2\xi\, f(\xi)\, g^*(\xi) \tag{89}$$

and

$$\int d^2\alpha\, \tilde{f}(\alpha)\, \tilde{g}(-\alpha) = \int d^2\xi\, f(\xi)\, g(\xi), \tag{90}$$

which apply also to operator-valued functions provided that complex conjugation is replaced by hermitian conjugation.

We may use the formula (84) for the delta function to derive a fermionic analog of the convolution theorem:

$$\begin{aligned}\int d^2\xi\, e^{\alpha\xi^* - \xi\alpha^*} f(\xi) g(\xi) &= \int d^2\beta\, d^2\xi\, e^{(\alpha-\beta)\xi^* - \xi(\alpha^*-\beta^*)} f(\xi) \int d^2\eta e^{\beta\eta^* - \eta\beta^*} g(\eta) \\ &= \int d^2\beta\, \tilde{f}(\alpha - \beta)\, \tilde{g}(\beta),\end{aligned} \tag{91}$$

which expresses the Fourier transform of the product of the two functions $f(\xi)$ and $g(\xi)$ as the convolution of their Fourier transforms $\tilde{f}(\alpha - \beta)$ and $\tilde{g}(\beta)$.

By using the normally ordered form (16) of the displacement operator, the eigenvalue property of the coherent states, and the preceding formula (84) for the delta function we find

$$\begin{aligned}\int d^2\gamma\, \langle\gamma|D(\alpha)|\gamma\rangle &= \int d^2\gamma\, \langle\gamma| e^{a^\dagger\alpha}\, e^{-\alpha^* a}|\gamma\rangle\, e^{\frac{1}{2}\alpha\alpha^*} \\ &= \int d^2\gamma\, e^{\gamma^*\alpha - \alpha^*\gamma + \frac{1}{2}\alpha\alpha^*} \\ &= \delta(\alpha) e^{\frac{1}{2}\alpha\alpha^*} = \delta(\alpha).\end{aligned} \tag{92}$$

The addition rule (19) for successive displacements now implies that for the multi-mode case

$$\int d^2\boldsymbol{\gamma}\, \langle\boldsymbol{\gamma}|D(\boldsymbol{\alpha})D(-\boldsymbol{\beta})|\boldsymbol{\gamma}\rangle = \delta(\boldsymbol{\alpha} - \boldsymbol{\beta}). \tag{93}$$

7 Operator Expansions

The preceding delta-function identity (93) and the completeness (55–58) of the displacement operators give us a means of expanding an arbitrary operator F in the form

$$F = \int d^2\boldsymbol{\xi} f(\boldsymbol{\xi}) D(-\boldsymbol{\xi}). \tag{94}$$

We may solve for the weight function $f(\boldsymbol{\xi})$ by multiplying on the right by the displacement operator $D(\boldsymbol{\alpha})$ and then taking the diagonal coherent-state matrix element in the state $|\boldsymbol{\beta}\rangle$ and integrating over $\boldsymbol{\beta}$:

$$\begin{aligned}\int d^2\boldsymbol{\beta}\,\langle\boldsymbol{\beta}|FD(\boldsymbol{\alpha})|\boldsymbol{\beta}\rangle &= \int d^2\boldsymbol{\beta}\int d^2\boldsymbol{\xi} f(\boldsymbol{\xi})\,\langle\boldsymbol{\beta}|D(-\boldsymbol{\xi})D(\boldsymbol{\alpha})|\boldsymbol{\beta}\rangle \\ &= \int d^2\boldsymbol{\xi} f(\boldsymbol{\xi})\,\delta(\boldsymbol{\alpha}-\boldsymbol{\xi}) = f(\boldsymbol{\alpha}). \qquad (95)\end{aligned}$$

The full expansion is thus

$$F = \int d^2\boldsymbol{\xi}\int d^2\boldsymbol{\beta}\,\langle\boldsymbol{\beta}|FD(\boldsymbol{\xi})|\boldsymbol{\beta}\rangle\, D(-\boldsymbol{\xi}). \qquad (96)$$

Such expansions will prove useful in the sections that follow.

The formula (84) for the delta function $\delta(\xi-\zeta)$ may be interpreted as a trace identity. From the eigenvalue property of the coherent states, it follows that

$$\delta(\xi-\zeta) = \int d^2\alpha\, e^{\alpha\xi^*-\xi\alpha^*}\, e^{\zeta\alpha^*-\alpha\zeta^*} = \int d^2\alpha\, e^{\alpha\xi^*-\xi\alpha^*}\,\langle\alpha|e^{\zeta a^\dagger}e^{-a\zeta^*}|\alpha\rangle \qquad (97)$$

in which we recognize the normally ordered form (16) of the displacement operator

$$\delta(\xi-\zeta) = \int d^2\alpha\, e^{\alpha\xi^*-\xi\alpha^*}\,\langle\alpha|D_N(\zeta)|\alpha\rangle. \qquad (98)$$

By using the trace formula (68), we may write this delta function as the trace

$$\begin{aligned}\delta(\xi-\zeta) &= \int d^2\alpha\, e^{\alpha\xi^*-\xi\alpha^*}\,\mathrm{Tr}\,[D_N(\zeta)|\alpha\rangle\langle-\alpha|] \\ &= \mathrm{Tr}\,[D_N(\zeta)E_A(-\xi)] \qquad (99)\end{aligned}$$

The subscript A has been chosen to indicate its anticipated of the product of the normally ordered displacement operator $D_N(\zeta)$ with an even operator $E_A(\xi)$ defined as the Fourier transform

$$E_A(\xi) = \int d^2\alpha\, e^{\xi\alpha^*-\alpha\xi^*}\,|\alpha\rangle\langle-\alpha| \qquad (100)$$

of the coherent-state dyadic $|\alpha\rangle\langle-\alpha|$. As intimated by its subscript, the operator $E_A(\xi)$ will turn out to be useful for dealing with anti-normally ordered operators.

We may now use the completeness (55–58) of the displacement operators and the trace identity (99) to expand an arbitrary operator F in terms of the normally ordered displacement operators $D_N(\xi)$,

$$F = \int d^2\xi \, f(\xi) D_N(-\xi). \tag{101}$$

We may solve for the function $f(\xi)$ by multiplying on the right by the operator $E_A(\zeta)$ and forming the trace:

$$\mathrm{Tr}\,[F E_A(\zeta)] = \int d^2\xi \, f(\xi) \mathrm{Tr}\,[D_N(-\xi) E_A(\zeta)] = \int d^2\xi \, f(\xi) \delta(\zeta - \xi) = f(\zeta). \tag{102}$$

The full expansion is thus

$$F = \int d^2\boldsymbol{\xi} \, \mathrm{Tr}\,[F E_A(\boldsymbol{\xi})] \; D_N(-\boldsymbol{\xi}). \tag{103}$$

By using the Grassmann calculus, one may compute the Fourier transform (100) of the coherent-state dyadic $|\alpha\rangle\langle-\alpha|$ and find for the operator $E_A(\xi)$ the formulas

$$\begin{aligned} E_A(\xi) &= |0\rangle\langle 0| - (\xi^* + a^\dagger)|0\rangle\langle 0|(\xi + a) && (104) \\ &= 2(\tfrac{1}{2} - a^\dagger a) + \xi\xi^* a a^\dagger + \xi a^\dagger - \xi^* a && (105) \end{aligned}$$

with which it is easy to exhibit the completeness of the operators $E_A(\xi)$:

$$\begin{aligned} I &= \int d^2\xi \; 2(1 + \xi^*\xi) \, E_A(\xi) && (106) \\ a &= \int d^2\xi \; (-\xi) \, E_A(\xi) && (107) \\ a^\dagger &= \int d^2\xi \; (-\xi^*) \, E_A(\xi) && (108) \\ \tfrac{1}{2} - a^\dagger a &= \int d^2\xi \; \tfrac{1}{2}\xi\xi^* \, E_A(\xi). && (109) \end{aligned}$$

Since the operators $E_A(\xi)$ are complete, we may expand an arbitrary operator G in terms of them,

$$G = \int d^2\xi \, g(\xi) E_A(-\xi) \tag{110}$$

and then use the trace formula (99) and the evenness of the displacement operators to evaluate the weight function $g(\xi)$,

$$\mathrm{Tr}\left[D_N(\zeta)G\right] = \int d^2\xi\, g(\xi)\, \mathrm{Tr}\left[D_N(\zeta)E_A(-\xi)\right] = \int d^2\xi\, g(\xi)\, \delta(\xi-\zeta) = g(\zeta). \tag{111}$$

The full expansion is thus

$$G = \int d^2\boldsymbol{\xi}\, \mathrm{Tr}\left[G D_N(\boldsymbol{\xi})\right]\; E_A(-\boldsymbol{\xi}). \tag{112}$$

8 Characteristic Functions

For a system described by the density operator ρ, we define the characteristic function $\chi(\boldsymbol{\xi})$ of Grassmann argument $\boldsymbol{\xi}$ (and $\boldsymbol{\xi}^*$) as the mean value

$$\chi(\boldsymbol{\xi}) = \mathrm{Tr}\left[\rho\, \exp\left(\sum_n \left(\xi_n a_n^\dagger - a_n \xi_n^*\right)\right)\right]. \tag{113}$$

It is thus a species of Fourier transform of the density operator ρ. Because $\xi_i^2 = \xi_i^{*2} = 0$, we may expand the exponential as

$$\chi(\boldsymbol{\xi}) = \mathrm{Tr}\left[\rho \prod_n \left(1 + \xi_n a_n^\dagger - a_n \xi_n^* + \xi_n^* \xi_n (a_n^\dagger a_n - \tfrac{1}{2})\right)\right]. \tag{114}$$

We may also define the normally ordered characteristic function $\chi_N(\boldsymbol{\xi})$ as

$$\chi_N(\boldsymbol{\xi}) = \mathrm{Tr}\left[\rho\, \exp\left(\sum_n \xi_n a_n^\dagger\right)\, \exp\left(-\sum_m a_m \xi_m^*\right)\right] \tag{115}$$

with the expansion

$$\chi_N(\boldsymbol{\xi}) = \mathrm{Tr}\left[\rho \prod_n \left(1 + \xi_n a_n^\dagger - a_n \xi_n^* + \xi_n^* \xi_n a_n^\dagger a_n\right)\right]. \tag{116}$$

The anti-normally ordered characteristic function $\chi_A(\boldsymbol{\xi})$ is

$$\begin{aligned} \chi_A(\boldsymbol{\xi}) &= \mathrm{Tr}\left[\rho \exp\left(-\sum_m a_m \xi_m^*\right) \exp\left(\sum_n \xi_n a_n^\dagger\right)\right] && (117) \\ &= \mathrm{Tr}\left[\rho \prod_n \left(1 + \xi_n a_n^\dagger - a_n \xi_n^* + \xi_n^* \xi_n (a_n^\dagger a_n - 1)\right)\right]. && (118) \end{aligned}$$

Because the density operator ρ is an even operator and because the displacement operators are constructed from bilinear forms in fermionic quantities, it follows that the characteristic functions are even in the sense that they commute with Grassmann variables.

8.1 The S-Ordered Characteristic Function

We may define a more general ordering of the annihilation operator a_n and the creation operator $a_n^\dagger$, much as we did earlier for boson-field operators [4]. It is an ordering specified by a real parameter s that runs from $s = -1$ for anti-normal ordering to $s = 1$ for normal ordering. For the quadratic case, the s-ordered product for fermions is

$$\{a_n^\dagger a_n\}_s = a_n^\dagger a_n + \tfrac{1}{2}(s-1), \tag{119}$$

to which we append the trivial definitions

$$\{a_n^\dagger\}_s = a_n^\dagger \quad \text{and} \quad \{a_n\}_s = a_n. \tag{120}$$

We note that the definition (119) differs by a crucial sign from that [4] of s-ordering for bosonic operators b_n and $b_n^\dagger$:

$$\{b_n^\dagger b_n\}_s = b_n^\dagger b_n + \tfrac{1}{2}(1-s). \tag{121}$$

In particular the anti-normally ordered product $\{a_n^\dagger a_n\}_{-1}$ is $-a_n a_n^\dagger$, and the symmetrically ordered product $\{a_n^\dagger a_n\}_0$ is half the commutator,

$$\{a_n^\dagger a_n\}_0 = \tfrac{1}{2}\left[a_n, a_n^\dagger\right]. \tag{122}$$

We define the s-ordered characteristic function $\chi(\boldsymbol{\xi}, s)$ as

$$\begin{aligned}
\chi(\boldsymbol{\xi}, s) &= \mathrm{Tr}\left[\rho\left\{\exp\left(\sum_n \left(\xi_n a_n^\dagger - a_n \xi_n^*\right)\right)\right\}_s\right] & (123)\\
&= \mathrm{Tr}\left[\rho \prod_n \left(1 + \xi_n a_n^\dagger - a_n \xi_n^* + \xi_n^* \xi_n \left\{a_n^\dagger a_n\right\}_s\right)\right] & (124)\\
&= \mathrm{Tr}\left[\rho \prod_n \left(1 + \xi_n a_n^\dagger - a_n \xi_n^* + \xi_n^* \xi_n \left(a_n^\dagger a_n + \tfrac{1}{2}(s-1)\right)\right)\right] & (125)\\
&= \mathrm{Tr}\left[\rho \exp\left(\sum_n \left(\xi_n a_n^\dagger - a_n \xi_n^* + \frac{s}{2}\, \xi_n^* \xi_n\right)\right)\right] & (126)\\
&= \chi(\boldsymbol{\xi}) \exp\left(\frac{s}{2} \sum_n \xi_n^* \xi_n\right), & (127)
\end{aligned}$$

which, incidentally, shows it to be an even function.

A particularly useful example of these characteristic functions is the case of the anti-normally ordered function $\chi_A(\boldsymbol{\xi}) = \chi(\boldsymbol{\xi}, -1)$. We see by inserting the resolution (61) of the identity between the exponential functions in its definition (117) that

$$\chi(\boldsymbol{\xi}, -1) = \mathrm{Tr}\left[\rho \exp\left(-\sum_m \beta_m \xi_m^*\right) \int d^2\boldsymbol{\beta}\, |\boldsymbol{\beta}\rangle\langle\boldsymbol{\beta}| \exp\left(\sum_n \xi_n \beta_n^*\right)\right], \tag{128}$$

in which we have replaced the annihilation and creation operators by their eigenvalues in the coherent states. By using the trace formula (67), we find

$$\chi(\boldsymbol{\xi}, -1) = \int d^2\boldsymbol{\beta} \exp\left(\sum_n \left(\xi_n \beta_n^* - \beta_n \xi_n^*\right)\right) \langle\boldsymbol{\beta}|\rho| - \boldsymbol{\beta}\rangle, \tag{129}$$

which expresses the anti-normally ordered characteristic function $\chi(\boldsymbol{\xi}, -1)$ as the Fourier transform of the matrix element $\langle\boldsymbol{\beta}|\rho| - \boldsymbol{\beta}\rangle$.

If we define the s-ordered displacement operator $D(\boldsymbol{\xi}, s)$ as

$$D(\boldsymbol{\xi}, s) = \{D(\boldsymbol{\xi})\}_s = D(\boldsymbol{\xi}) \exp\left(\frac{s}{2} \sum_n \xi_n^* \xi_n\right), \tag{130}$$

then we may write the s-ordered characteristic function (127) as the trace

$$\chi(\boldsymbol{\xi}, s) = \mathrm{Tr}\left[\rho\, D(\boldsymbol{\xi}, s)\right]. \tag{131}$$

9 S-Ordered Expansions for Operators

A convenient extension of the definition of the operator $E_A(\xi)$ is

$$E(\boldsymbol{\xi}, s) \equiv E_A(\boldsymbol{\xi}) \exp\left(\frac{s+1}{2}\sum_n \xi_n^* \xi_n\right), \tag{132}$$

from which we note that

$$E_A(\boldsymbol{\xi}) = E(\boldsymbol{\xi}, -1). \tag{133}$$

This is one sense in which the operator $E_A(\xi)$ is related to anti-normal ordering.

By using the s-ordered operators $D(\xi, s)$ and $E(\xi, s)$, we may generalize the expansions (103) and (112) of the arbitrary operators F and G to

$$F = \int d^2\boldsymbol{\xi}\, \mathrm{Tr}\left[F E(\boldsymbol{\xi}, -s)\right]\, D(-\boldsymbol{\xi}, s) \tag{134}$$

$$G = \int d^2\boldsymbol{\xi}\, \mathrm{Tr}\left[G D(\boldsymbol{\xi}, -s)\right]\, E(-\boldsymbol{\xi}, s). \tag{135}$$

The obvious generalization

$$\delta(\boldsymbol{\xi} - \boldsymbol{\zeta}) = \mathrm{Tr}\left[D(\boldsymbol{\xi}, s) E(-\boldsymbol{\zeta}, -s)\right] \tag{136}$$

of the trace formula (99) then gives the trace of the product FG as

$$\mathrm{Tr}\left[F\, G\right] = \int d^2\boldsymbol{\xi}\, \mathrm{Tr}\left[F E(\boldsymbol{\xi}, -s)\right]\, \mathrm{Tr}\left[G D(-\boldsymbol{\xi}, s)\right]. \tag{137}$$

We may now use the second Parseval relation (90) to cast the expansions (134) and (135) into forms that will prove to be quite useful. First let us

define the complete sets of operators $\tilde{D}(\boldsymbol{\alpha}, s)$ and $\tilde{E}(\boldsymbol{\alpha}, s)$ as the Fourier transforms of the operators $D(\boldsymbol{\xi}, s)$ and $E(\boldsymbol{\xi}, s)$:

$$\tilde{D}(\boldsymbol{\alpha}, s) \equiv \int d^2\boldsymbol{\xi} \, \exp\left(\sum_n (\alpha_n \xi_n^* - \xi_n \alpha_n^*)\right) D(\boldsymbol{\xi}, s) \tag{138}$$

$$\tilde{E}(\boldsymbol{\alpha}, s) \equiv \int d^2\boldsymbol{\xi} \, \exp\left(\sum_n (\alpha_n \xi_n^* - \xi_n \alpha_n^*)\right) E(\boldsymbol{\xi}, s). \tag{139}$$

Next let us define the weight functions $F_E(\boldsymbol{\alpha}, -s)$ and $G_D(\boldsymbol{\alpha}, -s)$ as the Fourier transforms of the traces

$$F_E(\boldsymbol{\alpha}, -s) \equiv \int d^2\boldsymbol{\xi} \, \exp\left(\sum_n (\alpha_n \xi_n^* - \xi_n \alpha_n^*)\right) \mathrm{Tr}\left[F E(\boldsymbol{\xi}, -s)\right] \tag{140}$$

$$G_D(\boldsymbol{\alpha}, -s) \equiv \int d^2\boldsymbol{\xi} \, \exp\left(\sum_n (\alpha_n \xi_n^* - \xi_n \alpha_n^*)\right) \mathrm{Tr}\left[G D(\boldsymbol{\xi}, -s)\right]. \tag{141}$$

It follows then from the Parseval relation (90) and from the expansions (134) and (135) that the operators $\tilde{D}(\boldsymbol{\alpha}, s)$ and $\tilde{E}(\boldsymbol{\alpha}, s)$ form complete sets of operators and afford us the expansions

$$F = \int d^2\boldsymbol{\alpha} \, F_E(\boldsymbol{\alpha}, -s) \, \tilde{D}(\boldsymbol{\alpha}, s) \tag{142}$$

$$G = \int d^2\boldsymbol{\alpha} \, G_D(\boldsymbol{\alpha}, -s) \, \tilde{E}(\boldsymbol{\alpha}, s) \tag{143}$$

of the arbitrary operators F and G. Applying the Parseval relation (90) to the trace formula (137), we have the trace relation

$$\mathrm{Tr}\left[F\, G\right] = \int d^2\boldsymbol{\alpha} \, F_E(\boldsymbol{\alpha}, -s) \, G_D(\boldsymbol{\alpha}, s). \tag{144}$$

The operators $\tilde{E}(\boldsymbol{\alpha}, s)$ are particularly simple when $s = \pm 1$. It follows from the definitions (132) and (100) of the operators $\tilde{E}(\boldsymbol{\alpha}, s)$ and $E_A(\xi)$, and from the formula (84) for the delta function that the operator $\tilde{E}(\boldsymbol{\alpha}, -1)$ is just the coherent-state dyadic

$$\tilde{E}(\boldsymbol{\alpha}, -1) = |\boldsymbol{\alpha}\rangle\langle -\boldsymbol{\alpha}|. \tag{145}$$

Similarly, by using the definitions (132) and (100) and the Fourier-transform relation (48), one may write the operator $\tilde{E}(\boldsymbol{\alpha}, 1)$ as the integral

$$\tilde{E}(\boldsymbol{\alpha}, -1) = \int \prod_i \left(-d^2\beta_i\right) \, e^{-(\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot(\boldsymbol{\alpha}^*-\boldsymbol{\beta}^*)} \, |\boldsymbol{\beta}\rangle\langle -\boldsymbol{\beta}|. \tag{146}$$

By performing the integration and referring to the explicit formula (31), we may show that the operator $\tilde{E}(\boldsymbol{\alpha}, 1)$ is the dyadic of the eigenstates (27) of the creation operators $|\boldsymbol{\alpha}\rangle'$:

$$\tilde{E}(\boldsymbol{\alpha}, 1) = |\boldsymbol{\alpha}\rangle' \, ' \langle -\boldsymbol{\alpha}|. \tag{147}$$

10 Quasi-Probability Distributions

Among the most important of the foregoing expansions, is the expansion (135) when the operator G is the density operator ρ,

$$\rho = \int d^2\boldsymbol{\xi} \, \mathrm{Tr}\left[\rho D(\boldsymbol{\xi}, s)\right] \, E(-\boldsymbol{\xi}, -s), \tag{148}$$

in which case the trace is the s-ordered characteristic function $\chi(\boldsymbol{\xi}, s)$,

$$\rho = \int d^2\boldsymbol{\xi} \, \chi(\boldsymbol{\xi}, s) \, E(-\boldsymbol{\xi}, -s). \tag{149}$$

We may define the s-ordered quasi-probability distribution $W(\boldsymbol{\alpha}, s)$ as the Fourier transform of the s-ordered characteristic function $\chi(\boldsymbol{\xi}, s)$

$$W(\boldsymbol{\alpha}, s) = \int d^2\boldsymbol{\xi} \exp\left(\sum_n \left(\alpha_n \xi_n^* - \xi_n \alpha_n^*\right)\right) \, \chi(\boldsymbol{\xi}, s); \tag{150}$$

both $W(\boldsymbol{\alpha}, s)$ and $\chi(\boldsymbol{\xi}, s)$ are even functions. It follows now from the expansion (143) that the s-ordered quasi-probability distribution $W(\boldsymbol{\alpha}, s)$ is the weight function for the density operator ρ in the expansion

$$\rho = \int d^2\boldsymbol{\alpha} \, W(\boldsymbol{\alpha}, s) \, \tilde{E}(\boldsymbol{\alpha}, -s). \tag{151}$$

The functions $W(\boldsymbol{\alpha}, s)$ for different values of the order parameter s are intimately related to one another because the characteristic functions obey the identity

$$\chi(\boldsymbol{\xi}, s) = \exp\left(\frac{s}{2}\boldsymbol{\xi}^* \cdot \boldsymbol{\xi}\right) \chi(\boldsymbol{\xi}) = \exp\left(\frac{(s-t)}{2}\boldsymbol{\xi}^* \cdot \boldsymbol{\xi}\right) \chi(\boldsymbol{\xi}, t). \tag{152}$$

The function $W(\boldsymbol{\alpha}, s)$ is therefore the Fourier transform of the product of $\exp\left(\frac{(s-t)}{2}\boldsymbol{\xi}^* \cdot \boldsymbol{\xi}\right)$ with the characteristic function $\chi(\boldsymbol{\xi}, t)$

$$W(\boldsymbol{\alpha}, s) = \int d^2\boldsymbol{\xi} \, \exp\left(\sum_n (\alpha_n \xi_n^* - \xi_n \alpha_n^*)\right) \exp\left(\frac{(s-t)}{2}\boldsymbol{\xi}^* \cdot \boldsymbol{\xi}\right) \chi(\boldsymbol{\xi}, t). \tag{153}$$

The Fourier transform of the characteristic function $\chi(\boldsymbol{\xi}, t)$ is $W(\boldsymbol{\alpha}, t)$, while that of $\exp\left(\frac{(s-t)}{2}\boldsymbol{\xi}^* \cdot \boldsymbol{\xi}\right)$ according to Eq.(48) is

$$\int d^2\boldsymbol{\xi} \, e^{\sum_n (\gamma_n \xi_n^* - \xi_n \gamma_n^*)} \, e^{\frac{(s-t)}{2}\boldsymbol{\xi}^* \cdot \boldsymbol{\xi}} = \prod_n \left[\frac{(t-s)}{2} \, e^{\frac{2}{(t-s)}\gamma_n \gamma_n^*}\right]. \tag{154}$$

The convolution theorem (91) now gives $W(\boldsymbol{\alpha}, s)$ as

$$W(\boldsymbol{\alpha}, s) = \int \prod_j \left[\frac{(t-s)}{2} d^2\beta_j\right] \exp\left[\frac{2}{(t-s)} \sum_i (\alpha_i - \beta_i)(\alpha_i^* - \beta_i^*)\right] W(\boldsymbol{\beta}, t). \tag{155}$$

A useful example of $W(\boldsymbol{\alpha}, s)$ is the function

$$W(\boldsymbol{\alpha}, -1) = \int d^2\boldsymbol{\xi} \exp\left(\sum_n (\alpha_n \xi_n^* - \xi_n \alpha_n^*)\right) \chi(\boldsymbol{\xi}, -1), \tag{156}$$

which according to Eq.(129) is the Fourier transform

$$W(\boldsymbol{\alpha}, -1) = \int d^2\boldsymbol{\xi} \, d^2\boldsymbol{\beta} \, \exp\left[\sum_n \left((\alpha_n - \beta_n)\, \xi_n^* - \xi_n \,(\alpha_n^* - \beta_n^*)\right)\right] \langle \boldsymbol{\beta} | \rho | -\boldsymbol{\beta} \rangle. \tag{157}$$

By using the delta-function identity (84), we see that this expression reduces to

$$\begin{aligned} W(\boldsymbol{\alpha}, -1) &= \int d^2\boldsymbol{\beta} \, \delta(\boldsymbol{\alpha} - \boldsymbol{\beta}) \, \langle \boldsymbol{\beta} | \rho | -\boldsymbol{\beta} \rangle \\ &= \langle \boldsymbol{\alpha} | \rho | -\boldsymbol{\alpha} \rangle. \end{aligned} \tag{158}$$

This function is the fermionic analogue of the function $Q(\boldsymbol{\beta}) = \langle\boldsymbol{\beta}|\rho|\boldsymbol{\beta}\rangle$ which is often used to represent the density operator ρ in terms of the bosonic coherent states $|\boldsymbol{\beta}\rangle$. It is the weight function that gives the mean values of anti-normally ordered products of creation and annihilation operators in terms of integrals of the corresponding products of Grassmann numbers.

11 Mean Values of Operators

We shall here be concerned with computing the mean values of the products of s-ordered monomials

$$\prod_i \{\left(a_i^\dagger\right)^{n_i} a_i^{m_i}\}_s \tag{159}$$

in which the exponents n_i and m_i take the values 0 or 1. The ordering of the modes labelled by the index i is arbitrary but fixed. We shall show that we may express the mean values of such products of monomials as integrals of the s-ordered weight function $W(\boldsymbol{\alpha}, s)$ multiplied by the monomials in the same order. By using the definition (150) of $W(\boldsymbol{\alpha}, s)$, we may write these integrals in the form

$$\begin{aligned}&\int d^2\boldsymbol{\alpha} \prod_i (\alpha_i^*)^{n_i} \alpha_i^{m_i} W(\boldsymbol{\alpha}, s)\\ &= \int d^2\boldsymbol{\alpha} \prod_i (\alpha_i^*)^{n_i} \alpha_i^{m_i} \int d^2\boldsymbol{\xi} \exp\left(\sum_j \left(\alpha_j \xi_j^* - \xi_j \alpha_j^*\right)\right) \chi(\boldsymbol{\xi}, s).\end{aligned} \tag{160}$$

It is now easy to write the monomial as a multiple derivative,

$$\begin{aligned}&\int d^2\boldsymbol{\alpha} \prod_i (\alpha_i^*)^{n_i} \alpha_i^{m_i} W(\boldsymbol{\alpha}, s)\\ &= \int d^2\boldsymbol{\alpha}\, d^2\boldsymbol{\xi} \prod_i \left[\frac{\partial^{n_i}}{\partial(-\xi_i)^{n_i}} e^{-\xi_i \alpha_i^*} \frac{\partial^{m_i}}{\partial(-\xi_i^*)^{m_i}} e^{-\xi_i^* \alpha_i}\right] \chi(\boldsymbol{\xi}, s).\end{aligned} \tag{161}$$

On using our formula (52) for integration by parts, we have

$$\begin{aligned}&\int d^2\boldsymbol{\alpha} \prod_i (\alpha_i^*)^{n_i} \alpha_i^{m_i} W(\boldsymbol{\alpha}, s)\\ &= \int d^2\boldsymbol{\xi}\, d^2\boldsymbol{\alpha} \exp\left(\sum_j \left(\alpha_j \xi_j^* - \xi_j \alpha_j^*\right)\right) \prod_i \left[\frac{\partial^{n_i}}{\partial(\xi_i)^{n_i}} \frac{\partial^{m_i}}{\partial(\xi_i^*)^{m_i}}\right] \chi(\boldsymbol{\xi}, s)\end{aligned} \tag{162}$$

in which we recognize the delta-function formula (85) which gives

$$\begin{aligned}
&\int d^2\boldsymbol{\alpha} \prod_i (\alpha_i^*)^{n_i}\, \alpha_i^{m_i}\, W(\boldsymbol{\alpha}, s) \\
&= \int d^2\boldsymbol{\xi}\, \delta(\boldsymbol{\xi}) \prod_i \frac{\partial^{n_i}}{\partial(\xi_i)^{n_i}} \frac{\partial^{m_i}}{\partial(\xi_i^*)^{m_i}}\, \chi(\boldsymbol{\xi}, s) && (163) \\
&= \prod_i \frac{\partial^{n_i}}{\partial(\xi_i)^{n_i}} \frac{\partial^{m_i}}{\partial(\xi_i^*)^{m_i}}\, \chi(\boldsymbol{\xi}, s) \Bigg|_{\boldsymbol{\xi}=0} && (164) \\
&= \mathrm{Tr}\left[\rho \prod_i \frac{\partial^{n_i}}{\partial(\xi_i)^{n_i}} \frac{\partial^{m_i}}{\partial(\xi_i^*)^{m_i}} \left(1 + \xi_i a_i^\dagger + \xi_i^* a_i + \xi_i^* \xi_i \,\{a_i^\dagger a_i\}_s\right)\Bigg|_{\boldsymbol{\xi}=0}\right] && (165)
\end{aligned}$$

If we recall the definitions of s-ordering in Eqs.(119) and (120), we then find

$$\int d^2\boldsymbol{\alpha} \prod_i (\alpha_i^*)^{n_i}\, \alpha_i^{m_i}\, W(\boldsymbol{\alpha}, s) = \mathrm{Tr}\left[\rho \prod_i \{\left(a_i^\dagger\right)^{n_i}\ a_i^{m_i}\}_s\right]. \tag{166}$$

In particular, by taking $n_i = m_i = 0$, we see that the weight function $W(\boldsymbol{\alpha}, s)$ is normalized,

$$\int d^2\boldsymbol{\alpha}\, W(\boldsymbol{\alpha}, s) = \mathrm{Tr}\rho = 1. \tag{167}$$

12 The P-Representation

Of the representations (151) for the density operator ρ, by far the most important is the one for $s = 1$ with the normally ordered weight function $P(\boldsymbol{\alpha}) = W(\boldsymbol{\alpha}, 1)$. By Eq.(145) it takes the simple form

$$\rho = \int d^2\boldsymbol{\alpha}\, P(\boldsymbol{\alpha})\, |\boldsymbol{\alpha}\rangle\langle -\boldsymbol{\alpha}|, \tag{168}$$

which recalls the P representation [2]–[3] for boson fields. Since the function $P(\boldsymbol{\alpha})$ is even, we may also write

$$\rho = \int d^2\boldsymbol{\alpha}\, P(\boldsymbol{\alpha})\, | -\boldsymbol{\alpha}\rangle\langle \boldsymbol{\alpha}|. \tag{169}$$

Because Grassmann integration is differentiation, the fermionic P representation is not affected by the mathematical limitations [2]–[5] that restricted somewhat the use of the bosonic P representation.

The P-representation may be used directly to compute the mean values of normally ordered products

$$
\begin{aligned}
\mathrm{Tr}\left(\rho\, a_k^{\dagger n} a_l^m\right) &= \int d^2\boldsymbol{\alpha} P(\boldsymbol{\alpha}) \langle \boldsymbol{\alpha} | a_k^{\dagger n} a_l^m | \boldsymbol{\alpha} \rangle \\
&= \int d^2\boldsymbol{\alpha} P(\boldsymbol{\alpha}) \alpha_k^{\dagger n} \alpha_l^m. \qquad (170)
\end{aligned}
$$

This extremely useful relation is just a special case of Eq.(166) for $s = 1$.

Since the operator $\tilde{E}(\boldsymbol{\alpha}, 1)$ is the dyadic (147) of the eigenstates of the creation operators, it follows from the expansion (151) that the weight function (158)

$$
Q(\boldsymbol{\alpha}) \equiv W(\boldsymbol{\alpha}, -1) = \langle \boldsymbol{\alpha} | \rho | -\boldsymbol{\alpha} \rangle \qquad (171)
$$

is the weight function in the representation

$$
\rho = \int d^2\boldsymbol{\alpha}\, Q(\boldsymbol{\alpha})\, |\boldsymbol{\alpha}\rangle' {}' \langle -\boldsymbol{\alpha}|, \qquad (172)
$$

which affords the simple way of computing the mean values of anti-normally ordered products that corresponds to Eq.(166) for $s = -1$.

Another use of the weight function $Q(\boldsymbol{\alpha}) = W(\boldsymbol{\alpha}, -1) = \langle \boldsymbol{\alpha} | \rho | -\boldsymbol{\alpha} \rangle$, however, is that it allows us to compute the weight function $P(\boldsymbol{\alpha}) = W(\boldsymbol{\alpha}, 1)$ of the P-representation as the simple convolution

$$
P(\boldsymbol{\alpha}) = \int \prod_m \left(-d^2\beta_m\right) \exp\left[-\sum_n \left(\alpha_n - \beta_n\right) \left(\alpha_n^* - \beta_n^*\right) \right] \langle \boldsymbol{\beta} | \rho | -\boldsymbol{\beta} \rangle, \quad (173)
$$

as follows from the general convolution formula (155) with $s = 1$ and $t = -1$. Although the analogous relation for bosons often is singular [2]–[5], this result holds for all fermionic density operators ρ.

13 Correlation Functions for Fermions

A principal use of the P representation for bosonic fields has been the evaluation of the normally ordered correlation functions, which play an important role in the theory of coherence and of the statistics of photon-counting experiments [2]. The analogously defined correlation functions for fields of fermionic atoms can be shown to play a similar role in the description of atom-counting experiments [9]. If we use $\psi(x)$ to denote the positive-frequency part of the Fermi field as a function of a space-time variable x, then the first two of these correlation functions may be defined as

$$G^{(1)}(x,y) = \mathrm{Tr}\left[\rho\psi^\dagger(x)\psi(y)\right] \tag{174}$$

$$G^{(2)}(x_1,x_2,y_2,y_1) = \mathrm{Tr}\left[\rho\psi^\dagger(x_1)\psi^\dagger(x_2)\psi(y_2)\psi(y_1)\right]. \tag{175}$$

$$\tag{176}$$

The nth-order correlation function is

$$G^{(n)}(x_1,\ldots,x_n,y_n,\ldots,y_1) = \mathrm{Tr}\left[\rho\psi^\dagger(x_1)\ldots\psi^\dagger(x_n)\psi(y_n)\ldots\psi(y_1)\right]. \tag{177}$$

If we expand the positive-frequency part of the Fermi field in terms of its mode functions $\phi_k(x)$ as

$$\psi(x) = \sum_k a_k\phi_k(x), \tag{178}$$

then its eigenvalue in the coherent state $|\boldsymbol{\alpha}\rangle$

$$\psi(x)|\boldsymbol{\alpha}\rangle = \varphi(x)|\boldsymbol{\alpha}\rangle \tag{179}$$

is the Grassmann field

$$\varphi(x) = \sum_k \alpha_k\phi_k(x) \tag{180}$$

in which the annihilation operators in (178) are replaced by the Grassmann variables $\boldsymbol{\alpha} = \{\alpha_k\}$.

We may use the P representation to evaluate the nth-order correlation function $G^{(n)}$ as the integral

$$G^{(n)}(x_1,\ldots,x_n,y_n,\ldots,y_1) \tag{181}$$

$$= \int d^2\boldsymbol{\alpha}\, P(\boldsymbol{\alpha}) \langle \boldsymbol{\alpha} | \psi^\dagger(x_1) \ldots \psi^\dagger(x_n) \psi(y_n) \ldots \psi(y_1) | \boldsymbol{\alpha} \rangle \tag{182}$$

$$= \int d^2\boldsymbol{\alpha}\, P(\boldsymbol{\alpha})\, \varphi^\dagger(x_1) \ldots \varphi^\dagger(x_n) \varphi(y_n) \ldots \varphi(y_1). \tag{183}$$

14 Chaotic States of the Fermion Field

The reduced density operator for a single mode of the fermion field can be represented by a 2×2 matrix for the states with occupation numbers 0 and 1. If the matrix is diagonal, it is specified completely by the mean number of quanta $\langle n \rangle$ in the mode. The density operator for the kth mode, in other words, must take the form

$$\rho_k = (1 - \langle n_k \rangle)\, |0\rangle\langle 0| + \langle n_k \rangle |1\rangle\langle 1|. \tag{184}$$

We shall speak of this density operator as representing a chaotic state of the kth mode. A chaotic state of the entire field will then be represented as a direct product of such density operators for all the modes of the field,

$$\rho_{ch} = \prod_k \rho_k. \tag{185}$$

It is specified by the complete set of mean occupation numbers $\{\langle n_k \rangle\}$.

The total number of fermions, $N = \sum_k a_k^\dagger a_k$, present in chaotic states will in general be indefinite. Indeed it is easily seen that in the state specified by Eq.(185) we have

$$\langle N^2 \rangle - \langle N \rangle^2 = \sum_k \langle n_k \rangle (1 - \langle n_k \rangle) \tag{186}$$

so that N can not be fixed unless all the $\langle n_k \rangle$ take the values 0 or 1. The indefiniteness of the number of particles present is a feature that the chaotic states of the fermion and boson fields have in common. For sufficiently large values of N, however, the fluctuations of $N/\langle N \rangle$ may be quite small so the specification of N in these relative terms may be quite precise. Fluctuations of this type in the number of particles present are a familiar property of the grand canonical ensemble in statistical mechanics, and that ensemble, as we shall see, represents a special class of chaotic states.

The single-mode density operator (184) can also be written as

$$\rho_k = (1 - \langle n_k \rangle) \left(\frac{\langle n_k \rangle}{1 - \langle n_k \rangle} \right)^{a_k^\dagger a_k} (|0\rangle\langle 0| + |1\rangle\langle 1|) \qquad (187)$$

in which we recognize the unit operator I_k for the subspace of the kth mode. Within this subspace we have

$$\rho_k = (1 - \langle n_k \rangle) \left(\frac{\langle n_k \rangle}{1 - \langle n_k \rangle} \right)^{a_k^\dagger a_k} . \qquad (188)$$

This expression can be used quite directly to evaluate the weight function $W(\boldsymbol{\alpha}, -1) = Q(\boldsymbol{\alpha})$.

We first note that for any real number v

$$v^{a^\dagger a} |\alpha\rangle = e^{\frac{1}{2}\alpha\alpha^*(1-v^2)} |\alpha v\rangle, \qquad (189)$$

so that we have

$$\begin{aligned} \langle \alpha | v^{a^\dagger a} | -\alpha \rangle &= e^{\frac{1}{2}\alpha\alpha^*(1-v^2)} \langle \alpha | -\alpha v \rangle \\ &= e^{\alpha\alpha^*(1+v)} . \end{aligned} \qquad (190)$$

Then if we let $v = \langle n_k \rangle / (1 - \langle n_k \rangle)$, we see that

$$\langle \alpha_k | \rho_k | -\alpha_k \rangle = (1 - \langle n_k \rangle) \exp \left(\frac{\alpha_k \alpha_k^*}{1 - \langle n_k \rangle} \right) \qquad (191)$$

and

$$\begin{aligned} Q(\boldsymbol{\alpha}) &= W(\boldsymbol{\alpha}, -1) = \prod_k \langle \alpha_k | \rho_k | -\alpha_k \rangle \\ &= \prod_k (1 - \langle n_k \rangle) \exp \left(\frac{\alpha_k \alpha_k^*}{1 - \langle n_k \rangle} \right) . \end{aligned} \qquad (192)$$

This product is the weight function appropriate to averaging anti-normally ordered operator products in chaotic states.

We may find the weight functions corresponding to all the other ordering schemes by using the convolution (155) with $t = -1$ and carrying out the

required integration with sufficient attention to the implicit minus signs. The result for the kth mode is

$$W_k(\alpha_k, s) = -\frac{s + 2\langle n_k \rangle - 1}{2} \exp\left(-\frac{2\alpha_k \alpha_k^*}{s + 2\langle n_k \rangle - 1}\right), \tag{193}$$

and the weight function for the multi-mode field is simply the product

$$W(\boldsymbol{\alpha}, s) = \prod_k W_k(\alpha_k, s). \tag{194}$$

Thus the function $W_k(\alpha_k, 0)$, which is analogous to the Wigner function for boson fields, is given by

$$W_k(\alpha_k, 0) = -\left(\langle n_k \rangle - \tfrac{1}{2}\right) \exp\left(-\frac{\alpha_k \alpha_k^*}{\langle n_k \rangle - \frac{1}{2}}\right), \tag{195}$$

and the function $W_k(\alpha_k, 1)$, which is the analogue of the function $P_k(\alpha_k)$ for boson fields, is

$$W_k(\alpha_k, 1) \equiv P_k(\alpha_k) = -\langle n_k \rangle \exp\left(-\frac{\alpha_k \alpha_k^*}{\langle n_k \rangle}\right). \tag{196}$$

The latter result is a particularly useful one since there are many physical contexts that call for the averaging of normally ordered products of annihilation and creation operators. For chaotic fields one may calculate all such averages as Grassmann integrals by making use of the fermionic P-representation with $P(\boldsymbol{\alpha})$ given by Eq.(196).

The minus signs in front of the expressions (195) and (196) may be somewhat surprising since these functions are the fermionic analogues of quasi-probability densities that are predominantly positive for boson fields. It is worth pointing out, therefore, that these signs result from our convention that defines $d^2\alpha$ as $d\alpha^* d\alpha$. Had we chosen the differential instead to be $d\alpha d\alpha^*$, the signs would have been positive.

For a chaotically excited boson field, the P-representation expresses the density operator as a gaussian integral of a diagonal coherent-state dyadic. For fermion fields the corresponding expression of ρ_k for a single mode is

$$\rho_k = -\langle n_k \rangle \int d^2\alpha_k \, e^{-\frac{\alpha_k \alpha_k^*}{\langle n_k \rangle}} \, |\alpha_k\rangle\langle -\alpha_k|. \tag{197}$$

According to Eq.(187), the density operator ρ_k can also be written as a sum over the m-fermion states as

$$\rho_k = (1 - \langle n_k \rangle) \sum_{m_k=0}^{1} \left(\frac{\langle n_k \rangle}{1 - \langle n_k \rangle} \right)^{m_k} |m_k\rangle\langle m_k|. \tag{198}$$

What we have shown, in effect, is that the two expressions are identical and that statistical averages can be evaluated by means of gaussian integrations for fermions as well as for bosons. The multi-mode density operator is represented, of course, by the product of the single-mode density operators, $\rho = \prod_k \rho_k$.

Fields in thermal equilibrium with a suitable particle reservoir represent particular examples of the kind of chaotic excitation we have been describing. If it is appropriate to describe such fields by means of the grand canonical ensemble, then their overall density operator may be written as

$$\rho = \frac{1}{\Xi(\beta, \mu)} \, e^{-\beta(H - \mu N)}, \tag{199}$$

where $\beta = 1/k_B T$, μ is the chemical potential, H is the hamiltonian for the system, N is the particle number, and the normalizing factor $\Xi(\beta, \mu)$ is the grand partition function. For a field with dynamically independent mode functions labeled by the index k, we can write

$$H = \sum_k \varepsilon_k a_k^\dagger a_k, \qquad N = \sum_k a_k^\dagger a_k \tag{200}$$

where ε_k is the energy of a particle in the kth mode.

Under these circumstances the equilibrium number of fermions in the kth mode is

$$\langle n_k \rangle = \frac{1}{e^{\beta(\varepsilon_k - \mu) + 1}}. \tag{201}$$

In that case the ratio $\langle n_k \rangle / (1 - \langle n_k \rangle)$ is simply the generalized Boltzmann factor

$$\frac{\langle n_k \rangle}{1 - \langle n_k \rangle} = e^{\beta(\varepsilon_k - \mu)}. \tag{202}$$

We then find that the product of the ρ_k given by Eq.(197) is precisely equal to the grand canonical density operator (199),

$$\int \prod_k \left(-\langle n_k \rangle d^2\alpha_k e^{-\frac{\alpha_k \alpha_k^*}{\langle n_k \rangle}} \right) |\boldsymbol{\alpha}\rangle\langle -\boldsymbol{\alpha}| = \frac{1}{\Xi(\beta, \mu)} \, e^{-\beta(H - \mu N)}. \tag{203}$$

There are many examples of thermal equilibria for which the P-representation on the left should furnish a useful computational tool.

15 Correlation Functions for Chaotic Field Excitations

We have introduced a succession of normally ordered correlation functions $G^{(n)}(x_1 \ldots x_n, y_n \ldots y_1)$ in section 13, and shown how they can be expressed as integrals over the Grassmann variables $\boldsymbol{\alpha} = \{\alpha_k\}$. For the case of chaotic fields, the appropriate weight function is

$$P(\boldsymbol{\alpha}) = \prod_k P_k(\alpha_k), \tag{204}$$

the product of the gaussian functions in Eq.(196). The first-order correlation function is thus given by

$$G^{(1)}(x,y) = \int \prod_k \left(-\langle n_k \rangle d^2\alpha_k e^{-\frac{\alpha_k \alpha_k^*}{\langle n_k \rangle}} \right) \langle \boldsymbol{\alpha} | \psi^\dagger(x) \psi(y) | \boldsymbol{\alpha} \rangle. \tag{205}$$

The fields ψ and $\psi^\dagger$ may now be replaced by their Grassmann field eigenvalues defined by (179) and (180). Their product is a quadratic form in the variables α_k and α_k^*, which is easily integrated:

$$\begin{aligned} G^{(1)}(x,y) &= \int \prod_k \left(-\langle n_k \rangle d^2\alpha_k e^{-\frac{\alpha_k \alpha_k^*}{\langle n_k \rangle}} \right) \sum_{l,m} \alpha_l^* \alpha_m \phi_l^\dagger(x) \phi_m(y) \\ &= \sum_k \langle n_k \rangle \, \phi_k^\dagger(x) \phi_k(y). \end{aligned} \tag{206}$$

To find the higher-order correlation functions, we can make use of a species of generating functional. We first define the Grassmann fields

$$\zeta(x) = \sum_k \beta_k \phi_k(x) \tag{207}$$

$$\eta(y) = \sum_k \gamma_k \phi_k(y), \tag{208}$$

and use them to construct the normally ordered expectation value

$$\Gamma[\zeta,\eta] \equiv \mathrm{Tr}\left[\rho \exp\left(\int \zeta(x)\psi^\dagger(x)d^4x\right)\exp\left(\int \psi(y)\eta^*(y)d^4y\right)\right]. \tag{209}$$

If we form the variational derivative of Γ with respect to $\zeta(x_1)$ from the left and with respect to $\eta^*(y_1)$ from the right, subsequently setting ζ and η to zero, then we find an alternative expression for the first-order correlation function,

$$\frac{\delta}{\delta_L\zeta(x_1)}\frac{\delta}{\delta_R\eta^*(y_1)}\Gamma|_{\zeta=\eta=0} = \mathrm{Tr}\left[\rho\psi^\dagger(x_1)\psi(y_1)\right] = G^{(1)}(x_1,y_1), \tag{210}$$

where left and right differentiation have been indicated explicitly in the subscripts.

It is evident then that one may generate all of the higher-order correlation functions by performing further differentiations,

$$G^{(n)}(x_1\ldots x_n, y_n\ldots y_1) = \frac{\delta}{\delta_L\zeta(x_1)}\cdots\frac{\delta}{\delta_L\zeta(x_n)}\frac{\delta}{\delta_R\eta^*(y_n)}\cdots\frac{\delta}{\delta_R\eta^*(y_1)}\Gamma|_{\zeta=\eta=0}\,. \tag{211}$$

To evaluate the generating functional Γ for a chaotic field, we make use of the orthonormality of the mode functions ϕ_k and then carry out the Grassmann integration

$$\begin{aligned}
\Gamma &= \int\prod_k\left(-\langle n_k\rangle d^2\alpha_k e^{-\frac{\alpha_k\alpha_k^*}{\langle n_k\rangle}}\right)\exp\left(\sum_l(\beta_l\alpha_l^*+\alpha_l\gamma_l^*)\right)\\
&= \prod_k(1+\langle n_k\rangle\beta_k\gamma_k^*) = \exp\left(\sum_k\langle n_k\rangle\beta_k\gamma_k^*\right)\\
&= \exp\left(\int\zeta(x)G^{(1)}(x,y)\eta^*(y)d^4xd^4y\right).
\end{aligned} \tag{212}$$

If we begin performing the variational differentiations to find the second-order correlation function, we may write

$$\frac{\delta}{\delta_R\eta^*(y_2)}\frac{\delta}{\delta_R\eta^*(y_1)}\Gamma|_{\eta=0} = \int\zeta(x)G^{(1)}(x,y_2)d^4x\int\zeta(x')G^{(1)}(x',y_1)d^4x'. \tag{213}$$

We then find

$$G^{(2)}(x_1x_2y_2y_1) = \frac{\delta}{\delta_L\zeta(x_1)}\frac{\delta}{\delta_L\zeta(x_2)}\int\zeta(x)G^{(1)}(x,y_2)d^4x\int\zeta(x')G^{(1)}(x',y_1)d^4x' \tag{214}$$

and since $\zeta(x)$ and $\zeta(x')$ anti-commute,

$$G^{(2)}(x_1,x_2,y_2,y_1) = G^{(1)}(x_1,y_1)G^{(1)}(x_2,y_2) - G^{(1)}(x_1,y_2)G^{(1)}(x_2,y_1). \tag{215}$$

The generalization to nth order is immediate. It expresses the nth-order correlation function for chaotic fields as a sum of products of first-order correlation functions with permuted arguments,

$$G^{(n)}(x_1\dots x_n,y_n\dots y_1) = \sum_P(-1)^P\prod_{j=1}^{n}G^{(1)}(x_j,y_{Pj}). \tag{216}$$

This expression is summed over the $n!$ permutations of the indices $1\dots n$. The factor $(-1)^P$ is the parity of the permutation, and the index Pj is the index that replaces j in the permutation.

The expression of the nth-order correlation function in terms of first-order correlation functions is characteristic of chaotic fields. Such fields are completely specified by the set of mean occupation numbers $\langle n_k\rangle$, and these are already contained in the first-order correlation function.

16 Fermion Counting Experiments

The use of photon counting techniques has for many years been the most direct means of investigating the statistical properties of light beams. Experiments of this type began with that of Hanbury Brown and Twiss [8] in 1956 and expanded greatly in scope with the development of the laser. The theory [3] underlying these experiments is based on the evaluation of quantum-mechanical expectation values of normally ordered products of electromagnetic field operators. The coherent states of the field [2] thus play a special role in the formulation of that theory. The application of the theory, furthermore, extends to boson fields of much more general sorts, including for example beams of heavy atoms [9].

In the case of the electromagnetic field, it has been shown [3] that the probability of detecting n photons in a given interval of time can be expressed as the nth derivative with respect to a parameter λ of a certain generating function $\mathcal{Q}(\lambda)$,

$$p(n) = \frac{(-1)^n}{n!} \frac{d^n}{d\lambda^n} \left. \mathcal{Q}(\lambda) \right|_{\lambda=1} . \tag{217}$$

The generating function $\mathcal{Q}(\lambda)$ for the electromagnetic field is the expectation value of a normally ordered exponential function of the form

$$\mathcal{Q}(\lambda) = \mathrm{Tr}\left(\rho{:}e^{-\lambda\mathcal{I}}{:}\right), \tag{218}$$

in which the symbols : : stand for normal ordering, and the operator $\mathcal{I}$ is a space-time integral of the product of the positive-frequency and negative-frequency parts of the field, $E^{(+)}$ and $E^{(-)}$, respectively.

For the case of fermion fields, it can easily be shown [9] that the probability of counting n fermions in a given interval of time falls into precisely the same general form. In the simplest instance, for detectors that respond to the density rather than the flux of the particles, the integral $\mathcal{I}$ takes the form

$$\mathcal{I} = \kappa \int \psi^\dagger(\vec{r}, t)\psi(\vec{r}, t) d^3r dt, \tag{219}$$

where the constant κ is a measure of the sensitivity of the counter and the integration is carried out over the counting-time interval and over the volume being observed.

To obtain the expectation value of the normally ordered exponential function in Eq.(218), then we may use the P-representation for the density operator ρ. In that case the field operators $\psi(\vec{r}, t)$ and $\psi^\dagger(\vec{r}, t)$ are, in effect, always applied to their eigenstates, coherent states such as $|\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\alpha}|$. They can then be replaced by their Grassmann field eigenvalue functions defined by Eq.(180) and its adjoint, so that we have

$$\mathcal{Q}(\lambda) = \int d^2\boldsymbol{\alpha} P(\boldsymbol{\alpha}) e^{-\lambda\mathcal{J}}, \tag{220}$$

where

$$\mathcal{J} = \kappa \int \varphi^*(\vec{r}, t)\varphi(\vec{r}, t) d^3r dt. \tag{221}$$

The expression $\mathcal{J}$ is a quadratic form that we can write as

$$\mathcal{J} = \sum_{k,k'} \alpha_k^* B_{kk'} \alpha_{k'}, \tag{222}$$

so the evaluation of the generating function $\mathcal{Q}(\lambda)$ reduces to the calculation of the integral

$$\mathcal{Q}(\lambda) = \int d^2\boldsymbol{\alpha} P(\boldsymbol{\alpha}) \exp\left(-\lambda \sum_{k,k'} \alpha_k^* B_{kk'} \alpha_{k'}\right), \tag{223}$$

in which the normal ordering symbols are no longer necessary because of the simple anti-commutation properties of the Grassmann variables α_k.

For the case of the chaotic fields defined in section 14, this integral takes the form

$$\mathcal{Q}(\lambda) = \int \prod_k \left(-\langle n_k\rangle d^2\alpha_k e^{-\frac{\alpha_k \alpha_k^*}{\langle n_k\rangle}}\right) \exp\left(-\lambda \sum_{k,k'} \alpha_k^* B_{kk'} \alpha_{k'}\right). \tag{224}$$

If we define a new set of variables $\beta_k = \alpha_k/\sqrt{\langle n_k\rangle}$, we find according to the rule (46) that the integral can be written as

$$\mathcal{Q}(\lambda) = \int \prod_k (-d^2\beta_k) \exp\left(\sum_{k,k'} \beta_k^* \left(\delta_{kk'} - \lambda M_{kk'}\right) \beta_{k'}\right), \tag{225}$$

where the matrix M is

$$M_{kk'} = \sqrt{\langle n_k\rangle} B_{kk'} \sqrt{\langle n_{k'}\rangle}. \tag{226}$$

A unitary linear transformation on the variables β_k can then be used to diagonalize the quadratic form in brackets. If the eigenvalues of the matrix $1 - \lambda M$ are μ_l, then the integral is easily seen, according to the formula (48) for $\alpha = 0$, to be

$$\mathcal{Q}(\lambda) = \prod_l \mu_l = \det\left(1 - \lambda M\right). \tag{227}$$

This result may be used directly to find the various probabilities given by Eq.(217). It contrasts quite interestingly with the generating function for boson counting distributions, which with closely corresponding definitions takes the form [3]

$$\mathcal{Q}_B(\lambda) = \frac{1}{\det\left(1 + \lambda M\right)}. \tag{228}$$

17 Some Examples

17.1 The Vacuum State

For the density operator

$$\rho = |0 \ldots 0\rangle\langle 0 \ldots 0| \tag{229}$$

which represents the multi-mode vacuum state, the normally ordered characteristic function $\chi_N(\boldsymbol{\xi})$ is

$$\begin{aligned} \chi(\boldsymbol{\xi})_N &= \mathrm{Tr}\left[\rho \exp\left(\sum_n \xi_n a_n^\dagger\right) \exp\left(-\sum_n a_n \xi_n^*\right)\right] \\ &= \langle 0 \ldots 0| \exp\left(\sum_n \xi_n a_n^\dagger\right) \exp\left(-\sum_n a_n \xi_n^*\right) |0 \ldots 0\rangle = 1. \end{aligned} \tag{230}$$

The weight function of the P representation is then

$$P(\boldsymbol{\alpha}) = \int d^2\boldsymbol{\xi} \exp\left(\sum_i (\alpha_i \xi_i^* - \xi_i \alpha_i^*)\right) = \delta(\boldsymbol{\alpha}). \tag{231}$$

The mean values of the normally ordered products of creation and annihilation operators all vanish

$$\mathrm{Tr}\left[\rho \prod_i \left(a_i^\dagger\right)^{n_i} a_i^{m_i}\right] = \int d^2\boldsymbol{\alpha} \prod_i (\alpha_i^*)^{n_i} \alpha_i^{m_i} \delta(\boldsymbol{\alpha}) = 0 \tag{232}$$

except for the trace

$$\mathrm{Tr}\,[\rho] = \int d^2\boldsymbol{\alpha}\, \delta(\boldsymbol{\alpha}) = 1. \tag{233}$$

The general weight function $W(\alpha, s)$ of the vacuum is given by

$$W(\boldsymbol{\alpha}, s) = \frac{1}{2}(1-s) \exp\left(\frac{2\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}^*}{(1-s)}\right). \tag{234}$$

17.2 A Physical Two-Mode Density Operator

Let us consider the most general physical two-mode fermionic density operator

$$
\begin{aligned}
\rho &= r\,|00\rangle\langle 00| + u\,|10\rangle\langle 10| + v\,|01\rangle\langle 01| + w\,|10\rangle\langle 01| + w^*\,|01\rangle\langle 10| \\
&\quad + x\,|00\rangle\langle 11| + x^*\,|11\rangle\langle 00| + t\,|11\rangle\langle 11| \qquad (235)
\end{aligned}
$$

in which $|10\rangle = a_1^\dagger|00\rangle$, $|11\rangle = a_2^\dagger a_1^\dagger|00\rangle$, *etc.*, and the Latin letters r, t, u, and v represent non-negative real numbers, while x and w may be complex. The non-zero traces are

$$
\begin{aligned}
\mathrm{Tr}\rho &= r + u + v + t = 1 && (236)\\
\mathrm{Tr}\rho\, a_1^\dagger a_1 &= u + t && (237)\\
\mathrm{Tr}\rho\, a_2^\dagger a_2 &= v + t && (238)\\
\mathrm{Tr}\rho\, a_2^\dagger a_1 &= w && (239)\\
\mathrm{Tr}\rho\, a_1^\dagger a_2 &= w^* && (240)\\
\mathrm{Tr}\rho\, a_1 a_2 &= x^* && (241)\\
\mathrm{Tr}\rho\, a_1^\dagger a_2^\dagger &= -x && (242)\\
\mathrm{Tr}\rho\, a_2^\dagger a_1^\dagger\, a_1 a_2 &= t. && (243)
\end{aligned}
$$

If the fermion number N commutes with the density operator ρ, then $x = x^* = 0$.

The normally ordered characteristic function $\chi_N(\boldsymbol{\xi})$ is

$$
\begin{aligned}
\chi(\boldsymbol{\xi})_N &= \mathrm{Tr}\left[\rho\left(1 + \xi_1 a_1^\dagger - a_1\xi_1^* + \xi_1^*\xi_1 a_1^\dagger a_1\right)\left(1 + \xi_2 a_2^\dagger - a_2\xi_2^* + \xi_2^*\xi_2 a_2^\dagger a_2\right)\right] \\
&= 1 + w\xi_1^*\xi_2 + w^*\xi_2^*\xi_1 + (u+t)\xi_1^*\xi_1 + (v+t)\xi_2^*\xi_2 \\
&\quad + x\,\xi_1\xi_2 + x^*\,\xi_2^*\xi_1^* + t\,\xi_1^*\xi_1\xi_2^*\xi_2. \qquad (244)
\end{aligned}
$$

The weight function $W(\boldsymbol{\alpha}, 1)$ is the Fourier transform of the normally ordered characteristic function $\chi_N(\boldsymbol{\xi})$

$$
\begin{aligned}
W(\boldsymbol{\alpha}, 1) &= \int d^2\xi_1 d^2\xi_2\,(1 + \alpha_1\xi_1^* + \alpha_1^*\xi_1 + \alpha_1^*\alpha_1\xi_1^*\xi_1) \\
&\quad \times (1 + \alpha_2\xi_2^* + \alpha_2^*\xi_2 + \alpha_2^*\alpha_2\xi_2^*\xi_2) \\
&\quad \times [1 + w\,\xi_1^*\xi_2 + w^*\,\xi_2^*\xi_1 + (u+t)\,\xi_1^*\xi_1 + (v+t)\,\xi_2^*\xi_2 \\
&\qquad + x\,\xi_1\xi_2 + x^*\,\xi_2^*\xi_1^* + t\,\xi_1^*\xi_1\xi_2^*\xi_2]\,,
\end{aligned}
$$

and after following the rules (38–40), we find

$$W(\boldsymbol{\alpha},1) = t + w\alpha_2\alpha_1^* + w^*\alpha_1\alpha_2^* + (v+t)\alpha_1^*\alpha_1 + (u+t)\alpha_2^*\alpha_2 + x\,\alpha_1\alpha_2 + x^*\,\alpha_2^*\alpha_1^* + \alpha_1^*\alpha_1\alpha_2^*\alpha_2. \quad (245)$$

We may now use this weight function to compute the mean values

$$\int d^2\alpha_1 d^2\alpha_2\, W(\boldsymbol{\alpha},1) = \mathrm{Tr}\rho = 1 \quad (246)$$

$$\int d^2\alpha_1 d^2\alpha_2\, \alpha_1^*\alpha_1\, W(\boldsymbol{\alpha},1) = \mathrm{Tr}\rho\, a_1^\dagger a_1 = u+t \quad (247)$$

$$\int d^2\alpha_1 d^2\alpha_2\, \alpha_2^*\alpha_2\, W(\boldsymbol{\alpha},1) = \mathrm{Tr}\rho\, a_2^\dagger a_2 = v+t \quad (248)$$

$$\int d^2\alpha_1 d^2\alpha_2\, \alpha_2^*\alpha_1\, W(\boldsymbol{\alpha},1) = \mathrm{Tr}\rho\, a_2^\dagger a_1 = w \quad (249)$$

$$\int d^2\alpha_1 d^2\alpha_2\, \alpha_1^*\alpha_2\, W(\boldsymbol{\alpha},1) = \mathrm{Tr}\rho\, a_1^\dagger a_2 = w^* \quad (250)$$

$$\int d^2\alpha_1 d^2\alpha_2\, \alpha_1\alpha_2\, W(\boldsymbol{\alpha},1) = \mathrm{Tr}\rho\, a_1 a_2 = x^* \quad (251)$$

$$\int d^2\alpha_1 d^2\alpha_2\, \alpha_1^*\alpha_2^*\, W(\boldsymbol{\alpha},1) = \mathrm{Tr}\rho\, a_1^\dagger a_2^\dagger = -x \quad (252)$$

$$\int d^2\alpha_1 d^2\alpha_2\, \alpha_2^*\alpha_1^*\alpha_1\alpha_2\, W(\boldsymbol{\alpha},1) = \mathrm{Tr}\rho\, a_2^\dagger a_1^\dagger\, a_1 a_2 = t \quad (253)$$

which agree with the results (236–243).

With $P(\boldsymbol{\alpha}) = W(\boldsymbol{\alpha},1)$ as given by (245), we may write the density operator (235) in the form of the fermionic P representation

$$\rho = \int d^2\boldsymbol{\alpha}\, P(\boldsymbol{\alpha})\, |-\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| = \int d^2\boldsymbol{\alpha}\, P(\boldsymbol{\alpha})\, |\boldsymbol{\alpha}\rangle\langle -\boldsymbol{\alpha}|. \quad (254)$$

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